Tapping wave energy through Longuet-Higgins microseism effect

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This paper is dedicated to the memory of Pierre Guével.

Introduction

It is well-known, since the works of Miche (1944) and Longuet-Higgins (1950), that, under a standing wave system, second-order pressures at twice the wave frequency penetrate the water column down to the sea-floor, whatever the waterdepth. Recently Guével proposed that energy could be extracted from the waves with a heaving horizontal plate at the sea bottom, located next to a reflective cliff or sea-wall, and tuned to oscillate at twice the wave frequency. Encouraging preliminary experiments were conducted in ACRI's wavetank (Lajoie *et al.* 2007).

In this paper we address the theoretical modeling of wave energy extraction with such a device, in the asymptotic case when the waterdepth is very large compared to the wavelength. In section I we assume that the first-order wave system is little modified, i.e. the power taken from the waves is a small portion of the power carried by the incoming wave. In section II we relieve this assumption and we show that one hundred percent of the wave power can be extracted, notwithstanding how large the waterdepth.

I. Classical perturbation theory

We assume a regular incoming wave system, with amplitude A and frequency ω, that is fully reflected from a vertical wall in $x = 0$. We use a coordinate system Oxz with $-\infty < x \leq 0$; $z = 0$ the unperturbed free surface. The waterdepth is h and we assume $kh \gg 1$ where $k = \omega^2/g$ is the wave number.

We make use of potential flow theory and we look for the velocity potential $\Phi(x, z, t)$ under the form

$$\Phi(x,z,t) = \epsilon\, \Phi^{(1)} + \epsilon^2\, \Phi^{(2)} + \epsilon^3\, \Phi^{(3)} + \ldots \tag{1}$$

with the small parameter ϵ identified with the wave steepness kA.

At first order the free surface elevation is

$$\eta^{(1)}(x,t) = A\ \cos(kx - \omega t) + A\ \cos(-kx - \omega t) = 2\,A\ \cos kx\ \cos\omega t \tag{2}$$

and the velocity potential

$$\Phi^{(1)}(x,z,t) = -\frac{2\,A\,g}{\omega}\ \mathrm{e}^{kz}\ \cos kx\ \sin\omega t = \Re\left\{-\frac{2\,\mathrm{i}\ A\,g}{\omega}\ \mathrm{e}^{kz}\ \cos kx\ \mathrm{e}^{-\mathrm{i}\,\omega t}\right\} = \Re\left\{\varphi^{(1)}(x,z)\ \mathrm{e}^{-\mathrm{i}\,\omega t}\right\} \tag{3}$$

The second-order velocity potential $\Phi^{(2)}$ satisfies the free surface equation, in $z = 0$

$$\Phi^{(2)}_{tt} + g\ \Phi^{(2)}_{z} = -\eta^{(1)}\ \frac{\partial}{\partial z}\,(\Phi^{(1)}_{tt} + g\ \Phi^{(1)}_{z}) - 2\ \nabla\Phi^{(1)} \cdot \nabla\Phi^{(1)}_{t} = -4\,A^2\,\omega^3\ \sin 2\omega t \tag{4}$$

As a result, $\Phi^{(2)}$ is simply

$$\Phi^{(2)} = A^2\,\omega\ \sin 2\omega t \tag{5}$$

The associated pressure $-\rho\ \Phi^{(2)}_t = -2\,\rho\,A^2\,\omega^2\ \cos 2\omega t$ is independent of the space coordinates.

At the foot of the reflective wall, a heaving plate with length $l \ll h$ is thus subjected to the load

$$F^{(2)} = 2\,\rho\,A^2\,\omega^2\,l\ \cos 2\omega t \tag{6}$$

The vertical velocity of the plate being

$$V^{(2)} = \Re\left\{v^{(2)}\ \mathrm{e}^{-2\mathrm{i}\,\omega t}\right\} = \Re\left\{\frac{q^{(2)}}{l}\ \mathrm{e}^{-2\mathrm{i}\,\omega t}\right\} \tag{7}$$

with $v^{(2)}$ and $q^{(2)}$ complex quantities, the time-averaged power absorbed by the plate is

$$P = \rho\, A^2\, \omega^2\, \Re\left\{q^{(2)}\right\} \tag{8}$$

From the free surface, the plate can be viewed as a point source located in $x = 0$, $z = -h$. Moreover, since $kh \gg 1$, the associated velocity potential $\Psi^{(2)}(x, z, t)$ approximately satisfies an homogeneous Dirichlet condition at the free surface. Therefore it writes

$$\Psi^{(2)} = \Re\left\{\psi^{(2)}\, \mathrm{e}^{-2\mathrm{i}\,\omega t}\right\} = \frac{1}{2\pi}\, \Re\left\{q^{(2)}\, \mathrm{e}^{-2\mathrm{i}\,\omega t}\right\}\, \ln \frac{x^2 + (z+h)^2}{x^2 + (h-z)^2} \tag{9}$$

and the associated vertical velocity at the free surface is

$$\psi_z^{(2)}(x, 0) = \frac{2\, q^{(2)}}{\pi}\, \frac{h}{x^2 + h^2} \tag{10}$$

We can now proceed to evaluating the third-order modification $\Phi^{(3)}$ of the velocity potential, due to interactions between $\Phi^{(1)}$ and $\Psi^{(2)}$. It satisfies the free surface equation

$$g\, \Phi_z^{(3)} + \Phi_{tt}^{(3)} = -\eta^{(1)}\, \Psi_{ttz}^{(2)} - 2\, \Phi_z^{(1)}\, \Psi_{zt}^{(2)} - 2\, \Phi_{zt}^{(1)}\, \Psi_z^{(2)} \tag{11}$$

Looking for the component at frequency ω

$$\Phi_1^{(3)}(x, z, t) = \Re\left\{\varphi_1^{(3)}(x, z)\, \mathrm{e}^{-\mathrm{i}\,\omega t}\right\} \tag{12}$$

we obtain

$$g\, \varphi_{1z}^{(3)} - \omega^2\, \varphi_1^{(3)} = -\mathrm{i}\, \omega\, k\, \psi_z^{(2)}\, {\varphi^{(1)}}^* \tag{13}$$

with * meaning the complex conjugate.

Replacing $\psi_z^{(2)}$ and $\varphi^{(1)}$ with their expressions (10) and (3) we obtain

$$\varphi_{1z}^{(3)} - k\, \varphi_1^{(3)} = -\frac{4\, A\, k\, h}{\pi}\, q^{(2)}\, \frac{\cos kx}{x^2 + h^2} = \alpha\, \frac{h\, \cos kx}{x^2 + h^2} \tag{14}$$

The right-hand side of the equation can be viewed as a pressure distribution acting over the free surface and generating a wave system. From Wehausen & Laitone (21.19) (with a correction in sign), we derive that the free surface elevation, away from the wall, takes the form

$$\eta_1^{(3)} = \Im\left\{\frac{\mathrm{i}\,\omega}{g}\, \mathrm{e}^{-\mathrm{i}\,kx - \mathrm{i}\,\omega t} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,ku}\, \alpha\, \frac{h\, \cos ku}{u^2 + h^2}\, \mathrm{d}u\right\} = \Im\left\{\frac{-2\mathrm{i}\,\omega k A}{g}\, q^{(2)}\, \mathrm{e}^{-\mathrm{i}\,kx - \mathrm{i}\,\omega t}\right\} \tag{15}$$

where, again, we have taken advantage that $kh \gg 1$.

This is to be added to the first-order reflected wave elevation $\eta_R^{(1)} = A\, \cos(kx + \omega t)$. Its amplitude A is therefore modified by $-2\omega\, kA\, \Re\left\{q^{(2)}\right\}/g$. The radiated power (or energy flux) $P = 1/2\, \rho\, g\, A^2\, C_G$ thus has been modified by the quantity

$$\Delta P = \frac{1}{2}\, \rho\, g\, \frac{\omega}{2k}\, \left[\left(A - \frac{2\omega k A}{g}\, \Re\left\{q^{(2)}\right\}\right)^2 - A^2\right] \tag{16}$$

that is, to the leading order

$$\Delta P = -\rho\, A^2\, \omega^2\, \Re\left\{q^{(2)}\right\} \tag{17}$$

in agreement with (8).

So we have made explicit the mechanism through which energy is being extracted from the waves: even though the plate is deeply submerged, the free surface heaves up and down at frequency 2ω and this heaving motion pumps energy from the waves. The larger the waterdepth the smaller the free surface motion amplitude but the larger the interaction area and the final result is the same.

These results are asymptotic in the sense that we have assumed that the first-order incoming and reflected wave systems are little modified by the energy transfer. In the following section we relieve this assumption.

II. Alternative approach

We make use of another small parameter, be μ, identified with $(kh)^{-1}$. The free surface elevation, at frequency ω, is now written as

$$\eta^{(1)}(x,t) = \Re\left\{\left[A_I(\mu x)\,\mathrm{e}^{\mathrm{i}\,kx} + A_R(\mu x)\,\mathrm{e}^{-\mathrm{i}\,kx}\right]\,\mathrm{e}^{-\mathrm{i}\,\omega t}\right\} \tag{18}$$

with the incoming A_I and reflected A_R amplitudes complex quantities.

The associated velocity potential is

$$\Phi^{(1)}(x,z,t) = \Re\left\{-\mathrm{i}\,\left[\frac{A_I(\mu x)\,g}{\omega}\,\mathrm{e}^{[k+\mu\,k_I(\mu x)]\,z}\,\mathrm{e}^{\mathrm{i}\,kx} + \frac{A_R(\mu x)\,g}{\omega}\,\mathrm{e}^{[k+\mu\,k_R(\mu x)]\,z}\,\mathrm{e}^{-\mathrm{i}\,kx}\right]\,\mathrm{e}^{-\mathrm{i}\,\omega t}\right\} \tag{19}$$

The Laplace equation, at order μ, gives

$$k_I\,A_I = -\mathrm{i}\,A_I' \qquad\qquad k_R\,A_R = \mathrm{i}\,A_R' \tag{20}$$

The associated second-order potential at frequency 2ω satisfies the Boundary Value Problem

$$\begin{aligned} \Delta\varphi^{(2)} &= 0 & -h \le z \le 0 \\ g\,\varphi_z^{(2)} - 4\,\omega^2\,\varphi^{(2)} &= -4\,\mathrm{i}\,A_I\,A_R\,\omega^3 & z = 0 \\ \varphi_z^{(2)} &= 0 & z = -h \end{aligned} \tag{21}$$

Neglecting the term $g\,\varphi_z^{(2)}$, of higher order in μ, in the free surface equation, we obtain the solution as

$$\varphi^{(2)}(x,z) = \frac{\mathrm{i}\,\omega}{2\pi}\int_{-\infty}^{\infty}\frac{\cosh\lambda(z+h)}{\cosh\lambda h}\,\mathrm{e}^{\mathrm{i}\,\lambda x}\,\mathrm{d}\lambda\int_{-\infty}^{\infty}A_I(u)\,A_R(u)\,\mathrm{e}^{-\mathrm{i}\,\lambda u}\,\mathrm{d}u \tag{22}$$

with $A_I\,A_R$ symmetrized with respect to $x = 0$. As a result the potential $\varphi^{(2)}$ at the foot of the reflective wall $(x = 0,\ z = -h)$ is given by

$$\begin{aligned} \varphi^{(2)}(0,-h) &= \frac{\mathrm{i}\,\omega}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda}{\cosh\lambda h}\int_{-\infty}^{\infty}A_I(u)\,A_R(u)\,\mathrm{e}^{-\mathrm{i}\,\lambda u}\,\mathrm{d}u \\ &= \frac{2\mathrm{i}\,\omega}{\pi}\int_{-\infty}^{0}A_I(u)\,A_R(u)\,\mathrm{d}u\int_0^{\infty}\frac{\cos\lambda u}{\cosh\lambda h}\,\mathrm{d}\lambda = \frac{\mathrm{i}\,\omega}{h}\int_{-\infty}^{0}\frac{A_I(u)\,A_R(u)}{\cosh\frac{\pi u}{2h}}\,\mathrm{d}u \end{aligned} \tag{23}$$

The velocity potential induced by the heaving motion of the plate, at the double frequency 2ω, is the same as in the previous section, with the associated vertical velocity at the free surface given by equation (10).

The right-hand side of equation (13) is now

$$-\mathrm{i}\,\omega\,k\,\psi_z^{(2)}\,\varphi^{(1)*} = \frac{2}{\pi}\,\omega^2\,q^{(2)}\,\frac{h}{x^2+h^2}\,\left\{A_I^*\,\mathrm{e}^{-\mathrm{i}\,kx} + A_R^*\,\mathrm{e}^{\mathrm{i}\,kx}\right\} \tag{24}$$

As for the left-hand side, to order one in μ, it is

$$g\,\varphi_z^{(1)} - \omega^2\,\varphi^{(1)} = -\mathrm{i}\,\frac{A_I\,g^2\,k_I}{\omega}\,\mathrm{e}^{\mathrm{i}\,kx} - \mathrm{i}\,\frac{A_R\,g^2\,k_R}{\omega}\,\mathrm{e}^{-\mathrm{i}\,kx} = -\frac{g^2}{\omega}\,A_I'\,\mathrm{e}^{\mathrm{i}\,kx} + \frac{g^2}{\omega}\,A_R'\,\mathrm{e}^{-\mathrm{i}\,kx} \tag{25}$$

Separating the complex amplitudes of the $\mathrm{e}^{\mathrm{i}\,kx}$ and $\mathrm{e}^{-\mathrm{i}\,kx}$ terms we get the coupled equations

$$\frac{\mathrm{d}A_I}{\mathrm{d}x} = -\frac{2\,\omega^3}{\pi g^2}\,q^{(2)}\,\frac{h}{x^2+h^2}\,A_R^* \qquad\qquad \frac{\mathrm{d}A_R}{\mathrm{d}x} = \frac{2\,\omega^3}{\pi g^2}\,q^{(2)}\,\frac{h}{x^2+h^2}\,A_I^* \tag{26}$$

together with the boundary conditions $A_I(-\infty) = A,\ A_R(0) = A_I(0)$, where A is the incoming wave amplitude at infinity. Setting $x = h\,X$ it comes

$$\frac{\mathrm{d}A_I}{\mathrm{d}X} = -\frac{2\,\omega^3}{\pi g^2}\,q^{(2)}\,\frac{1}{X^2+1}\,A_R^* \qquad\qquad \frac{\mathrm{d}A_R}{\mathrm{d}X} = \frac{2\,\omega^3}{\pi g^2}\,q^{(2)}\,\frac{1}{X^2+1}\,A_I^* \tag{27}$$

where the waterdepth h does not appear any longer.

It can be checked that the results of section I are recovered as the particular case when one assumes that the initial amplitudes are little modified. This amounts to stating that $A_R \simeq A_I \simeq A$ in the right-hand sides of equations (27). Then it comes immediately

$$A_I(0) - A = -\frac{2\,\omega^3\,A}{\pi g^2}\,q^{(2)}\,\int_{-\infty}^{0}\frac{\mathrm{d}X}{X^2+1} = -\frac{\omega^3\,A}{g^2}\,q^{(2)} \tag{28}$$

and

$$A_R(-\infty) = A_R(0) - \frac{\omega^3\,A}{g^2}\,q^{(2)} = A - \frac{2\,\omega^3\,A}{g^2}\,q^{(2)} \tag{29}$$

in agreement with equation (15).

Equations (27) can be rewritten, introducing $v = \arctan X$ ($v \in [-\pi/2\ \ 0]$) as the new variable

$$\frac{\mathrm{d}A_I}{\mathrm{d}v} = -\beta\,A_R^* \qquad\qquad \frac{\mathrm{d}A_R}{\mathrm{d}v} = \beta\,A_I^* \tag{30}$$

where $\beta = 2\,\omega^3\,q^{(2)}/(\pi\,g^2)$.

The solution, owing for $A_I(0) = A_R(0)$, is

$$A_I(v) \;=\; \beta\,c\;\cos|\beta|v - |\beta|\;c^*\;\sin|\beta|v \tag{31}$$

$$A_R(v) \;=\; \beta\,c\;\cos|\beta|v + |\beta|\;c^*\;\sin|\beta|v \tag{32}$$

with c obtained from

$$A_I(-\pi/2) = \beta\,c\;\cos\frac{|\beta|\,\pi}{2} + |\beta|\;c^*\;\sin\frac{|\beta|\,\pi}{2} = A \tag{33}$$

Conversely, equations (31) and (32) can be used to see under which conditions the reflected wave amplitude A_R can be annihilated at infinity. Stating $A_R(-\pi/2) = 0$ we get

$$\tan|\beta|\frac{\pi}{2} = \frac{\beta\,c}{|\beta|\,c^*} \tag{34}$$

or, designating by θ the argument of β and by ϕ the argument of c

$$\tan|\beta|\frac{\pi}{2} = \mathrm{e}^{\mathrm{i}\;(\theta+2\,\phi)} \tag{35}$$

Solutions are $|\beta_1| = 1/2,\ \theta + 2\,\phi = 0$; $|\beta_2| = 3/2,\ \theta + 2\,\phi = \pi$; $|\beta_3| = 5/2,\ \theta + 2\,\phi = 0$, etc.

Correlatively the amplitude $A = A_I(-\pi/2)$ of the incoming waves is $\sqrt{2}\,|\beta|\,|c|\,\mathrm{e}^{\mathrm{i}\;\theta/2}$ in the first case, $-\sqrt{2}\,|\beta|\,|c|\,\mathrm{e}^{\mathrm{i}\;(\pi/2-\theta/2)}$ in the second one, etc.

Equation (23) gives for the potential $\varphi^{(2)}$ at the foot of the wall

$$\varphi^{(2)}(0,-h) = \mathrm{i}\;\omega\,\beta_i^2\,c^2\int_{-\pi/2}^{0}\frac{\cos 2|\beta_i|v\;\;(1+\tan^2 v)}{\cosh\,(\pi/2\;\tan v)}\,\mathrm{d}v \tag{36}$$

Numerical evaluation of the integral gives 0.8119 for $|\beta_1| = 1/2$, 0.0109 for $|\beta_2| = 3/2$, -0.0675 for $|\beta_3| = 5/2$, etc. Since $\beta_i\,c^2$ is a real quantity (from (34)), it can be deduced that the load $-2\,\mathrm{i}\,\omega\,\rho\,\varphi^{(2)}(-h,0)$ and the velocity flux $q^{(2)}$ are in phase, meaning that the heaving plate must be in resonating condition. Retaining the first β_i value, for which the depth attenuation is lowest (and our assumptions of slowly-varying amplitudes best satisfied), the damping value of the heaving plate is finally obtained as

$$B_1 = 1.034\,\rho\,k^2\,A^2\,l^2\,\omega \tag{37}$$

proportional to the square of the wave steepness.

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