Scattering by arrays of bottom mounted cylinders and the approximation of near trapping in the time domain

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1 Introduction

Near-trapped modes appear in the frequency domain solution as large spikes in the response, and it is this context that they were considered by Evans & Porter (1997). However, in the time-domain the existence of a near trapped mode is associated with a slowly decaying mode which has a characteristic oscillation time and fall time. The solution in the time-domain was investigated by Eatock Taylor *et al.* (2006); Eatock Taylor & Meylan (2007) and the present work is a continuation of these. A near trapped mode is associated with a scattering frequency (also called resonance) close to the real axis. A scattering frequency is a pole of the analytic continuation of the scattering operator (or the resolvent). What this means is that the pole occurs for non-physical frequencies for which the scattered solution grows towards infinity (away from the scattering body). In practice, the scattering frequencies can usually be calculated by considering the formula for real values of frequencies but allowing the frequency to be complex and finding values for which the operator is not invertible (the approximation matrix is not invertible). In the context of water waves they have been investigated by Hazard & Lenoir (1993) and Hazard & Lenoir (2002) for the case of arbitrary two-dimensional bodies (although calculations were presented only for special cases) and Meylan (2002). Associated with the scattering frequencies is a mode, which is similar to an eigenfunction but grows to infinity. The method we use to find the modes associated with the scattering frequencies goes back to Steinberg (1968) and is used by Hazard & Loret (2008). The solution method in the time domain for arbitrary initial displacements (as opposed to a long crested incident wave packet) is based on the *generalised eigenfunction* method. This method goes back to the work of Povzner (1953); Ikebe (1960); Wilcox (1975) and have been used recently by Hazard & Lenoir (2002); Meylan (2002); Hazard & Loret (2007); Hazard & Meylan (2008).

2 Equations for N_c bottom mounted cylinders

We consider the case of N_c bottom mounted cylinders, radius a_l.centred at (x_l, y_l). in water of constant finite depth H. at $z = -H$. This gives us the following equation

$$\frac{J_m'(ka_l)}{H_m^{(1)\prime}(ka_l)}\left[\sum_{j=1,j\neq l}^{N_c}\sum_{\tau=-\infty}^{\infty} A_\tau^j H_{\tau-\nu}^{(1)}(kR_{jl})\mathrm{e}^{\mathrm{i}(\tau-m)\varphi_{jl}}\right] + A_m^l = -\frac{J_m'(ka_l)}{H_m^{(1)\prime}(ka_l)}D_m^l, \tag{1}$$

where (R_{jl}, φ_{jl}) are the polar coordinates of the mean centre position of cylinder l in the local coordinates of cylinder j. For the case of plane incident wave

$$D_\nu^l = e^{\mathrm{i}k(x_l\cos\chi + y_l\sin\chi)}\mathrm{e}^{\mathrm{i}\nu(\pi/2-\chi)}. \tag{2}$$

These equations follow from (Evans & Porter, 1997). For the case when the incident wave is cylindrical

$$D_\nu^l = \sum_{\nu=-\infty}^{\infty} J_{\nu-n}(kR_l)\mathrm{e}^{\mathrm{i}(\nu-n)(\pi-\vartheta_l)}. \tag{3}$$

3 Scattering Frequencies

We begin with the truncated version of equation (1) truncated to a finite number of modes which can be written as a matrix equation for the unknown vector $\mathbf{a}$ of coefficients A_ν^l :

$$\mathbf{M}(k)\,\mathbf{a} + \mathbf{a} = \mathbf{f}(k). \tag{4}$$

We find that the matrix $\mathbf{I}+\mathbf{M}\left(k\right)$ possesses zeros eigenvalues in the lower complex plane. These zeros are called scattering frequencies.

3.1 Calculation of the Residues

Suppose we have a scattering frequency at a complex wavenumber k_p. Near the point k_p it can be shown that (Steinberg (1968))

$$(\mathbf{I}+\mathbf{M}(k_p))^{-1}\approx\frac{\mathbf{u}_{k_p}\mathbf{u}^{*}_{k_p^*}}{\mathbf{u}^{*}_{k_p^*}\mathbf{M}^{(1)}\mathbf{u}_{k_p}\left(k-k_p\right)}$$

where $\mathbf{u}_{k_n}$ is the eigenvector with eigenvalue zero, $\mathbf{u}^{*}_{k_p^*}$, is the eigenvector of the adjoint operator with eigenvalue zero and $\mathbf{M}^{(1)}$ is the derivative of $\mathbf{M}$ at k_p. We can find the shape associated with each of the scattering frequencies which is given by the formula

$$U_{k_p}\left(\mathbf{x}\right)=\sum_{j=1,}^{N_c}\sum_{m=-N}^{N}u_m^jH_m^{(1)}(k_pr_j)\mathrm{e}^{\mathrm{i}m\theta_j}\qquad(5)$$

where u_m^j are the values corresponding to the eigenvector $\mathbf{u}_{k_0}$.

4 Time domain calculations

The solution in the frequency domain can be used to construct the solution in the time domain. This is well known for the case of a plane incident wave which is initially far from the body and can be calculated straight forwardly using the standard Fourier transform. However, when we consider an initial displacement which is nonzero around the cylinders the problem is of much greater complexity. The solution then requires the use of generalised eigenfunctions and the use of a special inner product. We introduce the operator $\mathbf{G}$ which maps the surface potential to the potential throughout the fluid domain. We find $\mathbf{G}\psi$ by solving

$$\Delta\Psi\left(\mathbf{x},z\right)=0,\quad\mathbf{x}\in\Omega,$$

$$\partial_n\Psi=0,\quad\mathbf{x}\in\partial\Omega,$$

$$\partial_n\Psi=0,\quad z=-H,$$

$$\Psi=\psi,\quad z=0,$$

and define $\mathbf{G}\psi=\Psi$. We introduce the operator $\partial_n\mathbf{G}$ which maps the surface potential to the normal derivative of potential at the surface (called the Dirichlet to Neumann map) which is given by

$$\partial_n\mathbf{G}\psi=\left.\partial_n\Psi\right|_{z=0}.$$

We can therefore write the time-dependent equations as the following

$$i\partial_t\begin{pmatrix}\phi\\-i\zeta\end{pmatrix}=\begin{pmatrix}0&1\\\partial_n\mathbf{G}&0\end{pmatrix}\begin{pmatrix}\phi\\-i\zeta\end{pmatrix}$$

We note that the evolution operator

$$\mathcal{A}=\begin{pmatrix}0&1\\\partial_n\mathbf{G}&0\end{pmatrix}$$

is self-adjoint in the Hilbert space given by the following inner product

$$\left\langle\begin{pmatrix}\phi\\-i\zeta\end{pmatrix},\begin{pmatrix}\psi\\-i\eta\end{pmatrix}\right\rangle_{\mathcal{H}}$$

$$=\int_{\Omega}\left(\nabla\mathbf{G}\phi\right)\left(\nabla\mathbf{G}\psi\right)^{*}d\Omega+\int_{\bar{\Omega}}\left(-i\zeta\right)\left(-i\eta\right)^{*}d\bar{\Omega}.$$

Note that this inner product is an expression for the energy.

4.1 Eigenfunctions of $\mathcal{A}$

The eigenfunctions are nothing more than frequency domain solutions and the frequency ω is exactly the eigenvalue. To actually calculate the eigenfunctions of $\mathcal{A}$ we need to specify the incident wave potential. The most natural form of the incident waves to consider is given by $J_n\left(kr\right)e^{in\theta}$. We write the eigenfunctions of $\mathcal{A}$ (with eigenvalue ω) in the vector form

$$\vec{\phi}_n(\mathbf{x},k\left(\omega\right))=\begin{pmatrix}\phi_n(\mathbf{x},k\left(\omega\right))\\-i\zeta_n(\mathbf{x},k\left(\omega\right))\end{pmatrix},\ n\in\mathbb{N}.$$

$\phi_n(\mathbf{x},k)$ are the solutions for an incident wave of the form $J_n\left(kr\right)e^{in\theta}$ and for ω positive are given by

$$\phi_n(\mathbf{x},k)=J_n\left(kr\right)e^{in\theta}+\sum_{j=1,}^{N_c}\sum_{m=-\infty}^{\infty}A_m^jH_m^{(1)}(kr_j)\mathrm{e}^{\mathrm{i}m\theta_j}$$

where A_m^j are found by solving equation (1) with D_m^l given by equation (3). We know that the eigenfunctions are orthogonal and the normalisation is given by

$$\left\langle\left(\vec{\phi}_n(\mathbf{x},k\left(\omega_n\right))\right),\vec{\phi}_m(\mathbf{x},k\left(\omega_m\right))\right\rangle_{\mathcal{H}}$$

$$=4\pi\frac{\omega_m^2}{k_n}\delta_{nm}\delta\left(\omega_n-\omega_m\right)\left.\frac{d\omega}{dk}\right|_{\omega=\omega_n}.$$

This normalisation is achieved by using the result that the eigenfunctions satisfy the same normalising condition with and without the scattering terms. This result, the proof of which is quite technical, has been shown for many different situations. The original proof was for Schrodinger's equation and was due to Povzner (1953); Ikebe (1960). A proof for the case of Helmholtz equation (our problem if the depth is shallow) was given by Wilcox (1975). Recently the proof was given for water waves by Hazard & Lenoir (2002); Hazard & Loret (2007).

The solution in the time domain on the free surface (it can be calculated throughout the fluid straightforwardly by including the depth dependence) is expanded in the waves $\vec{\phi}_n(\mathbf{x},k)$. This gives us

$$\begin{pmatrix} \Phi(\mathbf{x},t) \\ -i\zeta(\mathbf{x},t) \end{pmatrix} = \int_{-\infty}^{\infty} k \left\{ \sum_{n=-\infty}^{\infty} f_n(\omega)\, \vec{\phi}_n(\mathbf{x},k(\omega)) \right\} e^{-i\omega t} d\omega.$$

If we take the inner product we obtain

$$\left\langle \begin{pmatrix} \Phi(\mathbf{x},0) \\ -i\zeta(\mathbf{x},0) \end{pmatrix}, \vec{\phi}_n(\mathbf{x},k) \right\rangle_{\mathcal{H}} = 4\pi f_n(\omega)\,\omega^2 \frac{d\omega}{dk}.$$

This gives us the following expression for $f_n(\omega)$

$$f_n(\omega) = \frac{1}{4\pi\omega}\frac{dk}{d\omega}\int_{\bar{\Omega}} \left[\omega\Phi(\mathbf{x},0) + \zeta(\mathbf{x},0)\right] \times \left(\phi_n(\mathbf{x},k)\right)^* d\bar{\Omega}.$$

If $\Phi(\mathbf{x},0) = 0$ this expression for the displacement simplifies to

$$\zeta(\mathbf{x},t) = \int_{-\infty}^{\infty} |k| \Big\{ \sum_{n=-\infty}^{\infty} \Big(\frac{1}{4\pi} \times \int_{\Omega} \zeta(\mathbf{x},0)\left(\phi_n(\mathbf{x},k)\right)^* d\Omega\Big) \phi_n(\mathbf{x},k) \Big\} e^{-i\omega t} dk. \tag{6}$$

4.2 Approximation of the time domain solution

To keep the presentation simple, we consider the case where the initial potential is zero so that displacement is given by equation (6). We approximate

$$\phi_n(\mathbf{x},k) = \sum_{p=1}^{P} \left(\frac{\mathbf{u}_{k_0^*}^* \mathbf{f}_n(k)}{\mathbf{u}_{k_p^*}^* \mathbf{M}^{(1)} \mathbf{u}_{k_p}(k-k_p)} \right) U_{k_p}(\mathbf{x}),$$

which gives us

$$\zeta(\mathbf{x},t) \approx \mathrm{Re}\Big[\sum_{p=1}^{P} i\frac{k_0}{\left(k_p - k_p^*\right)} \sum_{n=-N}^{N} \left| \frac{\mathbf{u}_{k_0^*}^* \mathbf{f}_n(k_p)}{\mathbf{u}_{k_p^*}^* \mathbf{M}^{(1)} \mathbf{u}_{k_p}} \right|^2 \times \left(\int_{\Omega} \zeta(\mathbf{x},0) \left(U_{k_p}(\mathbf{x})\right)^* d\Omega \right) U_{k_p}(\mathbf{x})\, e^{-i\omega_p t}\Big],$$

where we have derived the last expression by closing the contour in the upper half plane and we have ignored the contribution from the branch cut.

4.3 Results for an arbitrary initial condition

For the purpose of calculations in this paper we use a grid of four cylinders as shown in Fig 1 There is one scattering frequency close to the real axis for the four cylinders at $2.7641 - 0.0122i$.

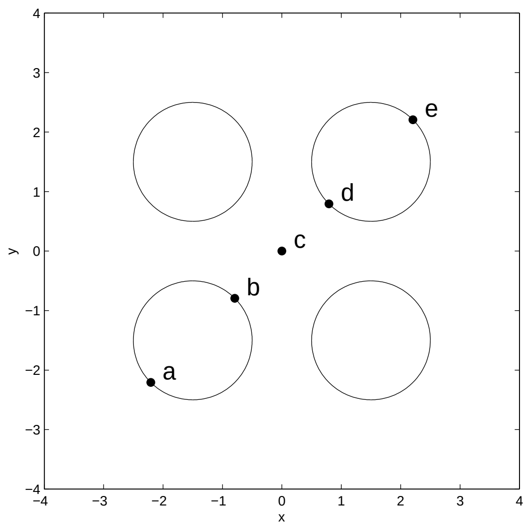

Fig. 1: The arrangement of cylinders and the points at which we will calculate the displacement.

We present results for an initial distribution of the form

$$\xi = e^{-2((x-0.5)^2+(y-0.5)^2)} + e^{-2((x+0.5)^2+(y+0.5)^2)} \tag{7}$$

The displacement is shown in Fig. 2 with the approximate solution shown as a dotted line. The true solution tends to the approximate solution after an initial period of time. The results in this figure are based on a shallow water approximation in which there is no dispersion so $\omega = k$. The effect of water depth is not marked for bottom mounted cylinders and is shown in Fig 3 (note that the time scales as $\sqrt{H}$ for shallow water with our non-dimensionalisation).

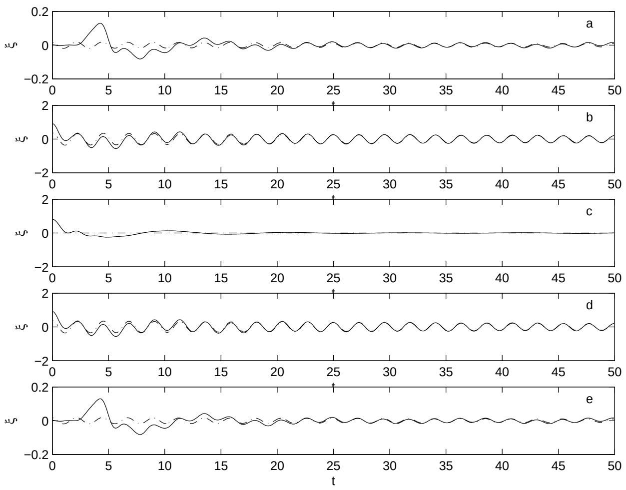

Fig. 2: The true (solid line) and approximate (dashed line) solution for the points a to e for an incident displacement given by equation (7).

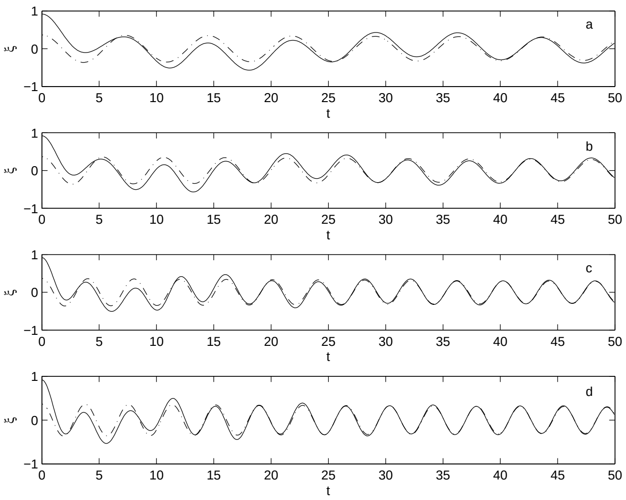

Fig. 3: The true (solid line) and approximate (dashed line) solution for the point b for an incident displacement given by equation (7) The water depth is given by 0.1 (a), 0.2 (b), 0.5 (c), and 1 (d).

5 Summary

We have considered arrays of vertical bottom mounted cylinders. This problem has been well studied and it is relatively simple to determine the single frequency solutions. We have shown that the solution in the time-domain can be calculated from the single frequency solutions, which requires a special formula that we have derived. It has previously been established that there is near trapping associated with a singularity in the analytic extension of the solution to the lower complex plane. We have shown here that there is also a mode associated with this zero which is very similar to an eigenfunction and we have shown how the contribution can be determined. We have also shown that, by deforming the contour of integration given by our expression for the time-dependent solution in terms of the single frequency solutions we can determine an approximate solution in terms of near trapped modes. The method outlined here would generalise easily to other situations, such as floating bodies. There is no requirement of near near trapped modes for the generalised eigenfunction expansion method. However, if the particular geometry does have near trapped modes, then the method outlined to find the approximation would work in a similar manner.

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