The effect of surface tension on localized free-surface oscillations about surface-piercing bodies

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Introduction

It is well known that non-unique solutions to the linear water-wave problem do exist—see McIver (1996) for the first example. In that paper symmetric pairs of surface-piercing bodies that support trapped modes were provided. However, surface tension was neglected in that analysis. In a recent paper by Harter *et al.* (2007), McIver's problem was updated to include surface tension, and it was shown that the qualitative nature of the surface-piercing bodies is unchanged. However, they neglected to include a condition describing the motion of the fluid at the contact line, arguing that such a condition would not have a significant effect for small values of surface tension. At present the contact-line problem is not well understood and there is no general agreement within the academic community as to what condition should be applied. For a discussion on some of the various contact-line conditions proposed, the reader is referred to Hocking (1987). In the present article, McIver's result will be extended to include surface tension whilst utilising a physically realistic contact-line condition. The condition is a limiting form of a more general relation proposed by Hocking (1987), and stipulates that the free surface is pinned to each contact point. This is an appropriate condition to apply when considering trapped modes as it does not result in attenuation of energy at the contact line.

Problem formulation and solution

We consider the inviscid, incompressible, irrotational motion of an unbounded fluid occupying a two-dimensional region $\mathscr{R}$, in the presence of a pair of symmetric surface-piercing bodies, the wetted boundaries of which we denote by $\mathscr{S}$. To this end, we introduce Cartesian co-ordinates (x, y) with the y-axis pointing vertically downwards; the undisturbed free surface $\mathscr{F}$ of the fluid lies along $y = 0$. In order to show that a non-unique solution exists, a non-trivial time-harmonic velocity potential $\Phi(x, y, t) = \mathrm{Re}\{\phi(x, y)\mathrm{e}^{-\mathrm{i}\omega t}\}$ must be found that satisfies

$$\nabla^2\phi = 0 \quad \text{in } \mathscr{R}, \tag{1}$$

$$\frac{\partial\phi}{\partial y} + (1+s)\phi - s\frac{\partial^3\phi}{\partial x^2\partial y} = 0 \quad \text{on } \mathscr{F}, \tag{2}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \mathscr{S}, \tag{3}$$

$$|\nabla\phi| \to 0 \quad \text{as } x^2 + y^2 \to \infty \ \ (y \geq 0). \tag{4}$$

This is a non-dimensional homogeneous boundary-value problem, and the parameter $s = \frac{Tk_0^2}{\rho g}$ is a Bond number which gives a measure of the relative importance of surface tension; here T is the surface tension of the fluid, k_0 is the wavenumber of the incident wave that satisfies the dimensional free surface condition, ρ is the fluid density and g is the acceleration due to gravity. When

$s \neq 0$ we also need to include a condition at each of the four lines of contact between $\mathscr{S}$ and $\mathscr{F}$, which we denote by $(\pm a, 0)$ and $(\pm b, 0)$ with $b > a$. We choose to apply the condition that the fluid surface is fixed, i.e.

$$\frac{\partial \phi}{\partial y}(\pm a, 0) = \frac{\partial \phi}{\partial y}(\pm b, 0) = 0. \tag{5}$$

It should be noted that this condition implies the streamlines are horizontal in the vicinity of each contact point.

The problem will be solved using an inverse procedure, with a combination of singularities placed on the free surface. The strengths and locations of these singularities are chosen in such a way that the conditions (1)–(5) are satisfied. The potential we will use to satisfy the problem is given by

$$\phi(x, y) = \int_0^\infty \frac{\mathrm{e}^{-my} \cos mx \, F(m)}{sm^3 + m - 1 - s} \mathrm{d}m, \tag{6}$$

where

$$F(m) = (1 + sm^2)(\cos m\xi + D \cos m\zeta) + A \cos ma + B \cos mb \tag{7}$$

with the parameters ξ, ζ, D, a, A, b and B to be found. This potential corresponds to singularities placed along the free surface at $x = \pm a$, $\pm\zeta$, $\pm\xi$ and $\pm b$, with strengths A, D, 1 and B respectively. The singularities at $(\pm\zeta, 0)$ and $(\pm\xi, 0)$ are wave sources, whereas the singularities positioned at $(\pm a, 0)$ and $(\pm b, 0)$ can be understood by noting that if we replace $F(m)$ by $\cos mx_0$ in (6), then the resultant potential $\bar{\phi}$ would satisfy

$$\frac{\partial \bar{\phi}}{\partial y} + (1 + s)\bar{\phi} - s\frac{\partial^3 \bar{\phi}}{\partial x^2 \partial y} = -\frac{\pi}{2}\big(\delta(x - x_0) + \delta(x + x_0)\big) \quad \text{on } y = 0. \tag{8}$$

To ensure a solution that does not exhibit singular behaviour in the fluid, we impose $a < \zeta < b$ and $a < \xi < b$. By construction (6) satisfies (1), (2) and is symmetric, and so we need only consider the region $x > 0$. We therefore have two edge conditions given by (5); one further condition comes from (4) which requires ϕ to be wave-free at large distances and so $F(1) = 0$. A further condition comes from the fact that we are thinking of the contact points $(a, 0)$ and $(b, 0)$ as lying on the surface of the same surface-piercing body, represented by the same streamfunction value. We now seek a set of parameters $(A, a, D, \zeta, \xi, B, b)$ that satisfy the aforementioned conditions. McIver's (1996) work for $s = 0$ corresponds to the choice $\xi = \frac{\pi}{2}$ and $A = B = 0$; consequently $D = 0$ (to satisfy the wave-free condition at infinity), and ζ is therefore irrelevant.

Results

We first consider the case $\xi = \frac{\pi}{2}$, as in McIver's (1996) solution, and for illustrative purposes set $\zeta = 1$. The streamline pattern for a few values of s is shown in figure 1. In order to produce similarly sized bodies, different values of a have been chosen for different s-values. For each value of surface tension s there is a maximum value of a, $a_{\max}$ say, above which surface-piercing bodies that separate the sources from the fluid cannot be found. This value increases as a increases, and it is found that as $s \to \infty$, $a_{\max} \approx 0.923$. It can be seen from figure 1 that as surface tension increases, the bodies become flatter at the free surface, in view of (5). Equivalently, as s gets smaller the region near each contact point in which the streamline slope is horizontal diminishes in size, and

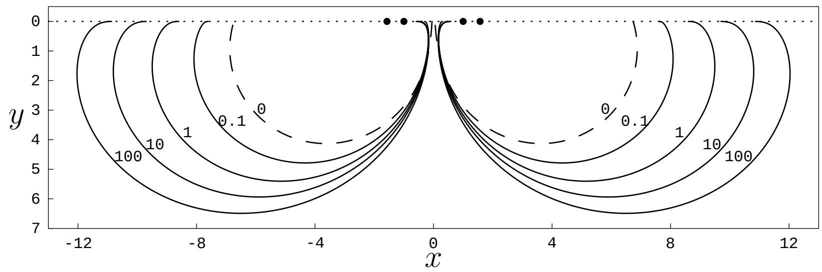

Figure 1: Typical pairs of surface-piercing bodies that support trapped modes, together with the sources that produce them. The values of s corresponding to each pair are shown in the figure.

it can be shown via asymptotic expansions that as $s \to 0$ these regions scale on $s^{\frac{1}{2}}$. Figure 2 illustrates a pair of streamlines that, due to the positions of the sources and the small value of s chosen, closely resemble those provided by McIver (1996). It can be seen however from the magnified plots that the streamline slope is still horizontal in a region close to each contact point, even though this is not visible on the larger plot. It can also be seen that additional closed loops of the same streamline exist near both contact points; however, these lie above the fluid and therefore have no physical significance.

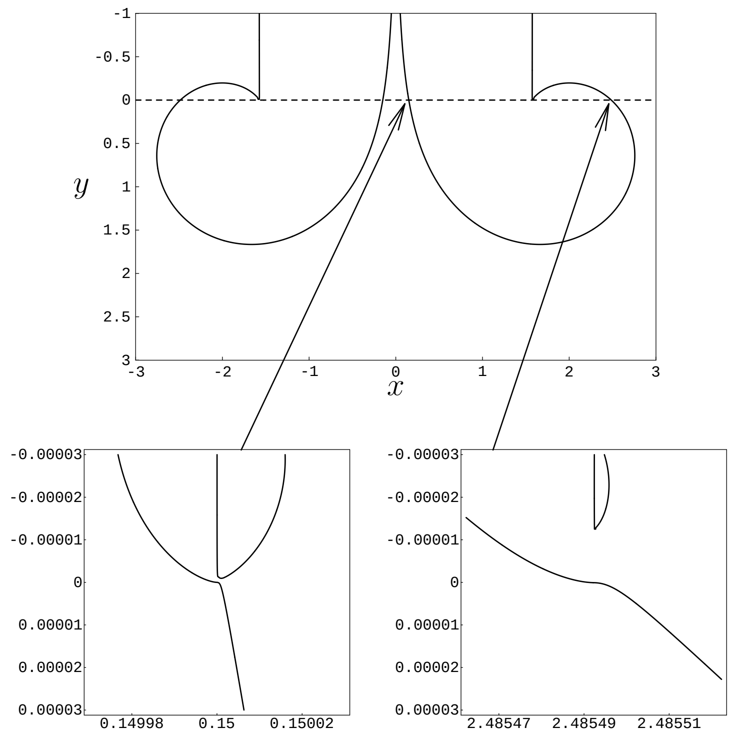

Figure 2: A pair of surface-piercing bodies that support trapped modes ($s = 10^{-10}$, $\xi = \frac{\pi}{2}$, $\zeta = 1$, $a = 0.15$, $b = 2.485492$), and magnified plots near the contact points.

The additional degrees of freedom obtained by introducing additional singularities to the velocity potential allow us to produce streamlines that are markedly different from those exhibited in McIver (1996). For example, by letting $\zeta \to \xi$ (i.e. by replacing the pair of sources with a single dipole) we are able to obtain bodies such as the ones shown in figure 3. These shapes closely resemble the submerged bodies presented in Harter *et al.* (2007). For certain values of ξ $(= \zeta)$, it is possible to obtain results where both contact points approach the dipole, and an example of this is shown in figure 3 (a). In general, the bodies can be made more elongated by reducing a; an example of this is given in figure 3 (b).

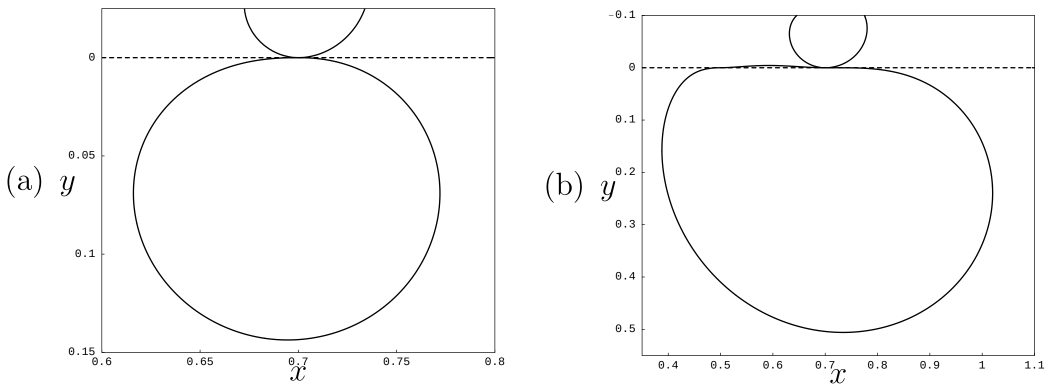

Figure 3: Surface-piercing bodies that, together with their mirror images, support trapped modes. (a) $s = 1 \times 10^{-15}$, $\xi = \zeta = 0.7$, $a = 0.699$, $b = 0.7001$; (b) $s = 0.01$, $\xi = \zeta = 0.7$, $a = 0.5$, $b = 0.713$.

The research described herein is presented in greater detail in Harter *et al.* (2008), where the authors also investigate surface tension effects in the case of zero-slope contact condition. It is found that trapped modes exist in this situation as well, and that the resulting streamlines are similar to those found by McIver.

Conclusion

The work of McIver (1996) has been updated to include surface tension whilst utilising a physically realistic contact-angle condition, relevant for the study of trapped modes. It has also been shown that this edge condition allows McIver's results to be recovered as $s \to 0$. Furthermore, it has been demonstrated that the results given in Harter *et al.* (2007) are valid, away from the contact points, for small values of s. Results are also obtained for large values of s; thus, utilising the pinned-edge contact-line condition, trapped modes can be found for all values of surface tension.

References

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