Scaling of wave-impact pressures in trapped air pockets

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1 Introduction

The determination of pressures generated by violent wave impacts is an important task in the design of coastal and offshore structures. Bagnold (1939) found that the highest pressures are often associated with entrapment of a small air pocket which is compressed during the impact. Improved understanding of the associated scale effects is needed, as the design of many structures is based, at least in part, on small-scale hydraulic model tests and compression of air does not follow the Froude law.

The complexity of the pressure variation within wave impacts has been highlighted by recent research which combined field and laboratory measurements at different scales (Bullock et al., 2007) with numerical modelling (Bredmose et al., 2004; Peregrine et al., 2006). Figure 1 (left) shows an example of a numerical computation for a wave which traps an air pocket. The numerical model solves the Euler equations for a mixture of water and ideal gas.

In validating the outcome of such investigations, one can often benefit from a comparison to a simpler test problem with known solution. This is the topic of the present paper. The study builds on the work of Bagnold (1939), who introduced a 1D piston model for analysing wave impacts with trapped air pockets, Mitsuyasu (1966), who derived an exact solution for the associated maximum pocket pressure and Lundgren (1969) who discussed the scaling implication of this result. Here we extend the derivation to the 2D and 3D axisymmetric cases. The 1D, 2D and 3D scaling laws provide insight into the physics of compressed air pockets and are closely related. This enables a single curve to be presented, for scaling impact pressures on the basis of measured pressures alone. Results from numerical computations of real wave impacts are in good agreement with this curve.

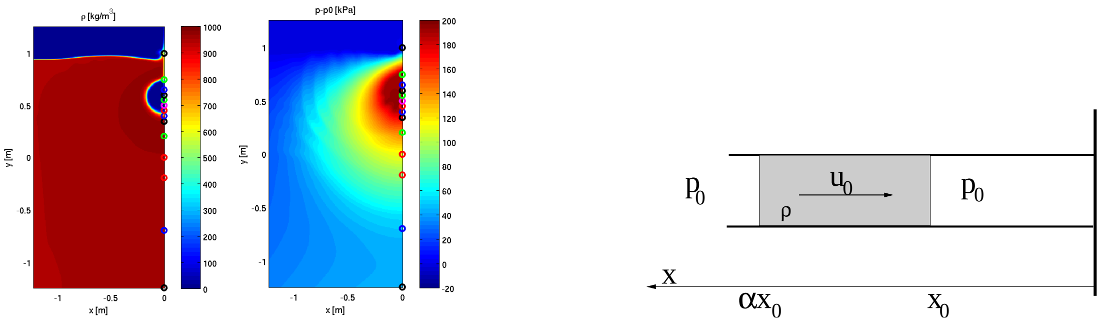

Figure 1: Left: Numerical example of the density and pressure fields in a wave impact with trapped air pocket. Right: Definition sketch for 1D piston model.

2 Scaling analysis

Consider the 1D piston model for a wave impact as sketched in figure 1 (right). An incompressible slug or 'piston' of water of density ρ compresses a pocket of air against a rigid wall at $x = 0$. Initially, the piston is travelling towards the wall with velocity u_0, with its front at $x = x_0$, its rear at $x = \alpha x_0$ and atmospheric pressure $p_0 = 10^5$ Pa on both sides.

We assume the air pressure to be uniform and the compression to be adiabatic in accordance with the ideal gas law. The pressure in the pocket is then

$$p = p_0 \left(\frac{x_0}{x}\right)^\gamma \tag{1}$$

where $\gamma = 1.4$ is the exponent of adiabatic compression. The work done, per unit width and height, for moving the piston to the position of maximum compression, x_{min}, is

$$W = \int_{x_0}^{x_{\text{min}}} p_0 \left(\frac{x_0}{x}\right)^\gamma - p_0 \quad \mathrm{d}(-x) = \frac{p_0 x_0}{\gamma - 1} \left[\left(\frac{x_{\text{min}}}{x_0}\right)^{\gamma-1} + (\gamma - 1)\frac{x_{\text{min}}}{x_0} - \gamma \right] \tag{2}$$

which by utilising (1) can be expressed in terms of the maximum pressure p_{max}

$$W = \frac{p_0 x_0}{\gamma - 1} \left[\left(\frac{p_{\text{max}}}{p_0}\right)^{(\gamma-1)/\gamma} + (\gamma - 1)\frac{p_{\text{max}}}{p_0} - \gamma \right]. \tag{3}$$

This work is equal to the initial kinetic energy of each unit width and height of the piston, $E_{\text{kin}} = (1/2)\rho(\alpha - 1)x_0 u_0^2$. Combining these two results yields the pressure law

$$\left(\frac{p_{\text{max}}}{p_0}\right)^{\frac{\gamma}{\gamma-1}} + (\gamma - 1)\left(\frac{p_{\text{max}}}{p_0}\right)^{-1/\gamma} - \gamma = \tfrac{1}{2}\frac{\gamma - 1}{p_0}\rho u_0^2(\alpha - 1). \tag{4}$$

Equation (4) is in agreement with the relationship derived by Mitsuyasu (1966). In Lundgren (1969), the right hand side is expressed in terms of a pseudo wave height. In the present setting, specific values of ρ, u_0 and α can be inserted in order to obtain the maximum pressure.

The implication of (4) with respect to scaling can be illustrated by means of an example. For $\rho = 1000$ kg/m^3, $u_0 = 4$ m/s and $\alpha = 1.4$, the right hand side of (4) takes the value $c = 1.3 \cdot 10^{-2}$, and a maximum pressure of $p/p_0 = 1.4$ can be found by iteration. This corresponds to a gauge pressure of $(p - p_0)/p_0 = 0.4$. In the context of wave impacts, the wave motion itself is Froude scalable. Thus for a geometric scale factor of S, the Froude scaled right hand side of (4) will take the value cS, as $u_0 \sim S^{1/2}$ Froude-wise. In figure 2, the maximum gauge pressure is plotted against cS for scaling factors in the range $S \in [1; 4096]$ (green curve).

The above example may be repeated using another set of values for ρ, u_0 and α. As a result of this, the right hand side of (4) can be written as $c_2 S$ where c_2 is a new constant. In the double logarithmic plot this will produce a curve identical to the one shown in figure 2, apart from a horisontal shift. Such a shift, however, does not affect the slope of the curve, which is still a unique function of $(p - p_0)/p_0$.

3 2D and 3D axisymmetric pockets

Although the 1D scaling law is instructive, its practical relevance is not entirely obvious. Wave impacts are seldomly 1D. Greater insight can be achieved by deriving the scaling laws for the 2D and 3D axisymmetric equivalents to the 1D system. Figure 3 shows the definition sketches. The 2D case describes a slug of water of constant volume that is compressing a pocket of air in a wedge shaped cavity. The instantaneous position of the front of the piston is denoted r and the instantaneous pressure of the trapped air is

$$p = p_0(r_0/r)^{2\gamma}. \tag{5}$$

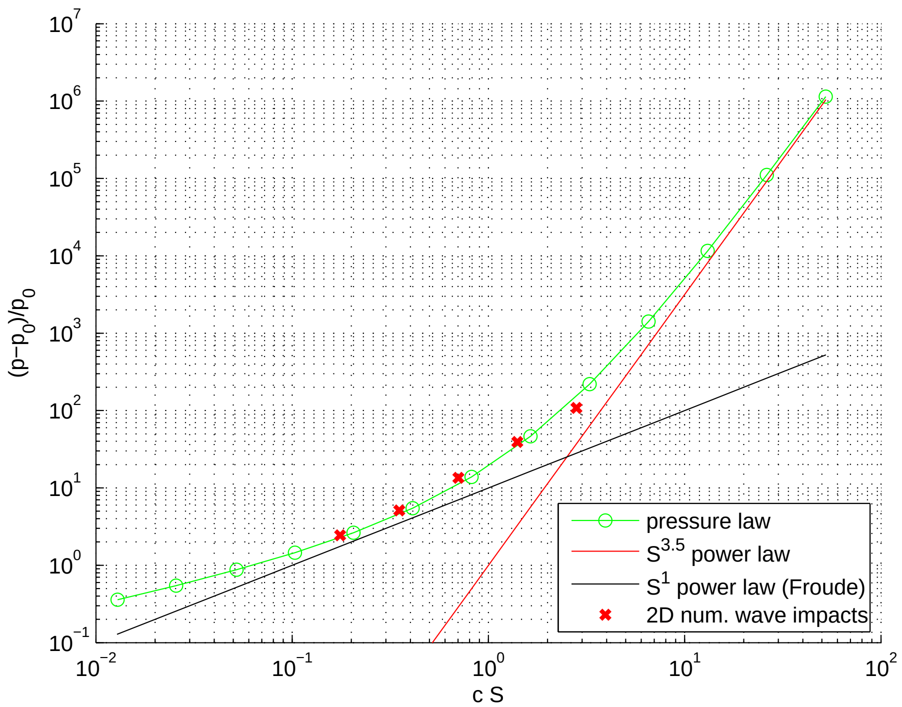

Figure 2: Maximum pressures of numerical computations of wave impacts with trapped air pockets. Comparison to the pressure law (4).

The initial velocity field is $u(r) = u_0 r_0 / r$ giving the following initial kinetic energy per unit with and angle

$$E_{\text{kin}} = \int_{r_0}^{\alpha r_0} \tfrac{1}{2}\rho \left(\frac{u_0 r_0}{r}\right)^2 r\mathrm{d}r = \tfrac{1}{2}\rho u_0^2 r_0^2 \ln \alpha. \tag{6}$$

As before, this energy equals the work for moving the slug of water to the position of maximum compression

$$W = \int_{r_0}^{r_m} p_0 \left[\left(\frac{r_0}{r}\right)^{2\gamma} - 1 \right] r\mathrm{d}(-r). \tag{7}$$

The final result, after combining (6), (7) and (5) is

$$\left(\frac{p_{\text{max}}}{p_0}\right)^{\frac{\gamma}{\gamma-1}} + (\gamma-1)\left(\frac{p_{\text{max}}}{p_0}\right)^{-1/\gamma} - \gamma = \frac{\gamma-1}{p_0}\rho u_0^2 \ln \alpha. \tag{8}$$

The 3D case can be analysed by similar means to obtain

$$\left(\frac{p_{\text{max}}}{p_0}\right)^{\frac{\gamma}{\gamma-1}} + (\gamma-1)\left(\frac{p_{\text{max}}}{p_0}\right)^{-1/\gamma} - \gamma = \tfrac{3}{2}\frac{\gamma-1}{p_0}\rho u_0^2 \left(1 - 1/\alpha\right). \tag{9}$$

We now see that the 1D, 2D and 3D pressure laws (4), (8) and (9) have identical left hand sides. Moreover, their right hand sides all scale proportionally to the geometric scaling factor S, when the problem dependent parameters are Froude scaled. Consequently, all three right hand sides can be written as $c_i S$, where c_i is a problem dependent constant. This means that for any choice of parameter values (α,ρ,u_0), or dimensions (1D, 2D or 3D), a log-log plot of $(p_{\text{max}} - p_0)/p_0$ against the right-hand side of the relevant pressure law will produce a curve identical to the green curve in figure 2, except for a horisontal shift caused by the different values of c_i. This shift, however, has no effect on the relationship between the slope of the curve and the pressure $(p - p_0)/p_0$.

Figure 3: Definition sketches for 2D and 3D axisymmetric air pockets.

4 Discussion

The relevance of the piston models to the more complex flows found in waves has been explored by means of numerical computations. The red crosses in figure 2 show the maximum pressures within the air pocket when the computation of figure 1 (left) was run at different scales. The numerical pressures show good agreement with the scaling curve.

The unique relationship between pressure and the slope of the scaling curve implies that any measured pressure can be scaled up or down by 1) reading off the corresponding value cS on the horizontal axis, 2) increasing or decreasing the value of cS by the desired scale factor and 3) using the new value of cS to read a scaled pressure from the curve.

For small scale factors, the local slope of the scaling curve provides a simple approximate scaling law. Inspection of (4) reveals that asymptotically, $(p - p_0)/p_0 \sim S^{\gamma/(\gamma-1)} = S^{3.5}$. This asymptotic relationship is plotted as a red line in figure 2. This relation, however, is only applicable for pressures exceeding 1000 MPa, and is thus not relevant in the context of wave impacts. Another line, representing $(p - p_0)/p_0 \sim S$, which corresponds to Froude scaling of the pocket gauge pressures, is plotted in the figure too. We see that the pressure curve takes this slope at $(p-p_0)/p_0 \approx 3$, i.e. for $p-p_0 \approx 300$ kPa. Hence the scaling of pressures of this size is well approximated by the Froude law. Most small scale laboratory pressures, however, are well below 300 kPa, and therefore must be expected to scale less than linearly with the scaling factor, if associated with a trapped air pocket. This may help to explain why Froude scaled laboratory pressures are often believed to be unrealistically high.

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