TIME-DEPENDENT HYDROELASTIC RESPONSE OF AN ELASTIC PLATE FLOATING ON SHALLOW WATER OF VARIABLE DEPTH

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INTRODUCTION

Studies on the behavior of large floating structures have been motivated by the design of platforms for various purposes. At present, there exists an extensive literature on hydroelastic analysis of the floating platforms [1]. In mathematical modelling, such platforms are often treated as thin elastic plates. Most authors assumed a flat seabed for their hydroelastic analysis of floating platforms. In reality, the seabed is not uniform in depth.

To our knowledge, the consideration of a varying water depth was made only for diffraction problem, by solving the linear hydroelastic problem for a single frequency [2-4]. A floating thin elastic plate on shallow water of variable depth is considered in this paper. This problem has been chosen because the sea-bottom effects become more significant in shallow water, than that in deep water (see, for example, [4)]. Proposed method may be used for any unsteady 2D problem of linear shallow-water theory, but here the motion of the elastic beam plate is considered for a travelling localized wave. The solution of this problem for a flat bottom was given in [5]. Unsteady response of an elastic beam floating on a shallow water of uniform depth under external load was considered in [6].

MATHEMATICAL FORMULATION

An elastic beam of width $2L$ floats on the surface of an inviscid incompressible fluid layer. The surface of the fluid that is not covered with the plate is free. The fluid region S is divided into three parts: S_1 $(|x| < L)$, S_2 $(x < -L)$, S_3 $(x > L)$, where x is the horizontal coordinate. Without the plate, the fluid depth is equal to $H(x)$ in S_1, and the fluid depths in the left and right hand domains of constant depth S_2 and S_3 are equal to H_1 and H_2, respectively. The fluid depth is assumed to be continuous, so that $H(-L) = H_1$, and $H(L) = H_2$. With the plate, the fluid depth in S_1 is equal to $h(x) = H(x) - d$, where d is the draft of the plate. It is assumed that the maximal depth of the fluid is small in comparison with the horizontal dimension of the plate, and the shallow water approximation is used. The velocity potentials describing the fluid motion in the regions S_j are denoted by $\phi_j(x,t)$ $(j = 1, 2, 3)$, where t is time.

A deflection of an elastic plate $w(x,t)$ is described by the equation:

$$D\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} + g\rho w + \rho\frac{\partial \phi_1}{\partial t} = 0 \quad (x \in S_1), \tag{1}$$

where D is the flexural rigidity of the plate; m is the mass per unit length of the plate; ρ is the fluid density, and g is the gravity acceleration. The draft of the plate is equal $d = m/\rho$.

According to linear shallow-water theory, the following relation is valid:

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial x}\left(h(x)\frac{\partial \phi_1}{\partial x}\right) \quad (x \in S_1). \tag{2}$$

In the free-water regions, the velocity potentials $\phi_2(x,t)$ and $\phi_3(x,t)$ satisfy the equations

$$\frac{\partial^2 \phi_2}{\partial t^2} = gH_1\frac{\partial^2 \phi_2}{\partial x^2} \quad (x \in S_2), \qquad \frac{\partial^2 \phi_3}{\partial t^2} = gH_2\frac{\partial^2 \phi_3}{\partial x^2} \quad (x \in S_3). \tag{3}$$

The displacements of the free surface $\eta_2(x,t)$ and $\eta_3(x,t)$ are determined in the regions S_2 and S_3 from the relations

$$\eta_j = -\frac{1}{g}\frac{\partial \phi_j}{\partial t} \quad (x \in S_j),\ j = 2, 3.$$

If $|x| = L$, the matching conditions (continuity of pressure and mass) should be satisfied:

$$\frac{\partial \phi_1}{\partial t} = \frac{\partial \phi_2}{\partial t},\ \frac{\partial \phi_1}{\partial x} = \frac{H_1}{h_1}\frac{\partial \phi_2}{\partial x}\ (x = -L), \quad \frac{\partial \phi_1}{\partial t} = \frac{\partial \phi_3}{\partial t},\ \frac{\partial \phi_1}{\partial x} = \frac{H_2}{h_2}\frac{\partial \phi_3}{\partial x}\ (x = L),\ h_{1,2} = H_{1,2} - d. \tag{4}$$

As it is noted that elevation of water surface is not continuous on the boundaries between region S_1 and regions S_2, S_3. At the edges of the beam, the free-edge conditions are satisfied, which imply that the bending moment and shear force are equal to zero: $\partial^2 w/\partial x^2 = \partial^3 w/\partial x^3 = 0$ at $|x| = L$.

It is assumed, that at the initial time the plate and fluid in the regions S_1 and S_3 are at rest. In region S_2, the localized displacement of the free surface $\eta_0(x - \sqrt{gH_1}t)$ travels to the right. The function $\eta_0(\xi)$ is different from zero only at $|\xi| < c$. At $t = 0$ the displacement reaches the left edge of the plate and the plate begins to undergo a complex bending motion in response to the incoming wave. Consequently, the initial conditions have the form:

$$w = \eta_3 = \frac{\partial \phi_1}{\partial t} = \frac{\partial \phi_3}{\partial t} = 0, \quad \eta_2 = \eta_0(x), \quad \frac{\partial \phi_2}{\partial t} = -g\eta_0(x) \quad (t = 0). \tag{5}$$

Non-dimensional variables are used below: L is taken as the length scale and $\sqrt{L/g}$ as the time scale.

MODE EXPANSIONS

The beam deflection is sought in the form of an expansion in the eigenfunctions of vibrations of a free-edges beam in vacuum

$$w(x,t) = \sum_{n=0}^{\infty} a_n(t) W_n(x). \tag{6}$$

Here the functions $a_n(t)$ are to be determined and the functions $W_n(x)$ are solutions of the spectral problem:

$$W_n^{(IV)} = \lambda_n^4 W_n \quad (|x| \le 1), \quad W_{2n}' = W_{2n+1} = 0 \quad (x = 0), \quad W_n'' = W_n''' = 0 \quad (|x| = 1).$$

The prime denotes differentiation with respect to x. These solutions have the form

$$W_0 = 1/\sqrt{2}, \quad W_{2n} = D_{2n}[\cos(\lambda_{2n}x) + S_{2n}\cosh(\lambda_{2n}x)],$$

$$W_1 = \sqrt{3/2}x, \quad W_{2n+1} = D_{2n+1}[\sin(\lambda_{2n+1}x) + S_{2n+1}\sinh(\lambda_{2n+1}x)],$$

where $S_n = \cos\lambda_n/\cosh\lambda_n$ and $D_n = 1/\sqrt{1 + (-1)^n S_n^2}$. The eigenvalues of λ_n are found from the equation $\tan\lambda_n + (-1)^n \tanh\lambda_n = 0 \quad (n \ge 2), \quad \lambda_0 = \lambda_1 = 0$. The functions $W_n(x)$ form a complete orthogonal system for which

$$\int_{-1}^{1} W_n(x) W_m(x) \mathrm{d}x = \delta_{mn},$$

where δ_{mn} is the Kroneker symbol.

We substitute expansion (6) into (1) and initial conditions (5), multiply the obtained relations by $W_m(x)$, and integrate them over x from -1 to 1. Using the properties of the functions $W_n(x)$, we obtain the set of ordinary differential equations (ODE's)

$$\gamma \ddot{a}_m + (\delta\lambda_m^4 + 1)a_m + f_m(t) = 0$$

with the initial conditions $a_m(0) = \dot{a}_m(0) = 0$, where

$$f_m(t) = \int_{-1}^{1} W_m \frac{\partial \phi_1}{\partial t} \mathrm{d}x, \quad \delta = \frac{D}{\rho g L^4}, \ \gamma = \frac{d}{L}, \tag{7}$$

and an overdot denotes differentiation with respect to time.

A solution for $\phi_1(x,t)$ is sought in the form

$$\phi_1(x,t) = \sum_{n=0}^{\infty} \dot{a}_n(t) \Psi_n(x) + q(x,t),$$

where the functions $\Psi_n(x)$ satisfy the equation

$$\Psi_n'(x) = -V_n(x)/h(x), \quad V_n'(x) = W_n(x)$$

and have the form

$$\Psi_n(x) = -\int_{-1}^{x} \frac{V_n(\xi)}{h(\xi)} \mathrm{d}\xi, \quad V_0 = \frac{x}{\sqrt{2}}, \quad V_1 = \frac{\sqrt{3}x^2}{2\sqrt{2}},$$

$$V_{2n} = \frac{D_{2n}}{\lambda_{2n}}[\sin(\lambda_{2n}x) + S_{2n}\cosh(\lambda_{2n}x)], \quad V_{2n+1} = \frac{D_{2n+1}}{\lambda_{2n+1}}[S_{2n+1}\cosh(\lambda_{2n+1}x) - \cos(\lambda_{2n+1}x)].$$

The function $q(x,t)$ is to be determined. According the Eq.(2) and the initial conditions (5), the function $q(x,t)$ has the form

$$q(x,t) = Q(x)u(t) + v(t), \quad Q(x) = \int_{-1}^{x} h^{-1}(\xi)\mathrm{d}\xi, \quad u(0) = v(0) = 0.$$

The functions $u(t)$ and $v(t)$ are determined from the matching conditions (4).

The solution for $\phi_2(x,t)$ is sought in the form

$$\phi_2(x,t) = \phi_0(x,t) + \psi(x,t), \tag{8}$$

where $\phi_0(x,t)$ is the velocity potential of incident wave and is determined from the equation $\partial\phi_0/\partial x = \eta_0/\sqrt{H_1}$. According to Eq.(3), the solution for $\psi(x,t)$ has the form

$$\psi(x,t) = \begin{cases} A((x+1)/\sqrt{H_1} + t), & -(1+\sqrt{H_1}t) < x < -1, \\ 0, & x < -(1+\sqrt{H_1}t), \end{cases}$$

where the function $A(\xi)$ is unknown and should be determined.

In a similar manner, we can seek the solution for $\phi_3(x,t)$

$$\phi_3(x,t) = \begin{cases} B(t - (x-1)/\sqrt{H_2}), & 1 < x < 1+\sqrt{H_2}t, \\ 0, & x > 1+\sqrt{H_2}t, \end{cases}$$

where the function $B(\xi)$ is to be determined.

Using the matching conditions (4), we have

$$\dot{A} = (u + \dot{a}_0R_0 - \dot{a}_1R_1)/\sqrt{H_1} - \alpha(t), \quad \dot{B} = (\dot{a}_0R_0 + \dot{a}_1R_1 - u)/\sqrt{H_2}. \tag{9}$$

Here $R_n = V_n(1),\ R_0 = 1/\sqrt{2},\ R_1 = \sqrt{1.5}/2,\ R_n = 0$ at $n \geq 2;\ \alpha(t) = \eta_0(-1,t)$.

The functions $f_m(t)$ in (7) have the form

$$f_m(t) = \sum_{n=0}^{\infty} \ddot{a}_n(t)(\Lambda_nR_m + C_{nm}) + (\beta R_m + \Lambda_m)\dot{u} + \sqrt{2}\delta_{m1}[(u + \dot{a}_0R_0 - \dot{a}_1R_1)/\sqrt{H_1} - 2\alpha(t)],$$

where

$$C_{nm} = \int_{-1}^{1} \frac{V_n(x)V_m(x)}{h(x)}\mathrm{d}x, \quad \Lambda_n = \Psi_n(1),\ \beta = Q(1).$$

The final set of ODE's has the form

$$\sum_{n=0}^{\infty} \ddot{a}_n(\gamma\delta_{nm} + \Lambda_nR_m + C_{nm}) + \sqrt{2}\delta_{m1}\left[\frac{1}{\sqrt{H_1}}(u + \dot{a}_0R_0 - \dot{a}_1R_1) - 2\alpha(t)\right] + (\delta\lambda_m^4 + 1)a_m + (\beta R_m + \Lambda_m)\dot{u} = 0,$$

$$\dot{u} = \frac{1}{\beta}\left[\frac{1}{\sqrt{H_2}}(\dot{a}_0R_0 + \dot{a}_1R_1 - u) + \frac{1}{\sqrt{H_1}}(\dot{a}_1R_1 - \dot{a}_0R_0 - u) - \sum_{n=0}^{\infty}\ddot{a}_n\Lambda_n + 2\alpha(t)\right].$$

Once the $a_n(t)$ and $u(t)$ are determined, we can find all characteristics of motion of the fluid and the elastic beam. For example, the displacement of the free surface of the fluid in region S_2 can be written in view of (8) as $\eta_2(x,t) = \eta_0(x,t) + \zeta(x,t)$, where

$$\zeta(x,t) = \begin{cases} -\dot{A}((x+1)/\sqrt{H_1} + t), & -(1+\sqrt{H_1}t) < x < -1, \\ 0, & x < -(1+\sqrt{H_1}t). \end{cases}$$

In region S_3, we have

$$\eta_3(x,t) = \begin{cases} -\dot{B}(t-(x-1)/\sqrt{H_2}), & 1 < x < 1+\sqrt{H_2}t, \\ 0, & x > 1+\sqrt{H_2}t. \end{cases}$$

The functions $\dot{A}(\xi)$ and $\dot{B}(\xi)$ are determined from (9).

ENERGY RELATION

Total energy of the incident wave E_0 is equal to

$$E_0 = \int_{-(1+2c)}^{-1} \eta_0^2(x)\mathrm{d}x.$$

This energy transfers to the oscillations of the elastic beam and reflected and transmitted surface waves. At $t \to \infty$, the beam oscillations decay and the beam returns to its original state. The energy of reflected wave motion $E_r(t)$ is

$$E_r(t) = \int_{-(1+\sqrt{H_1}t)}^{-1} \zeta^2(x,t)\mathrm{d}x = \sqrt{H_1}\int_0^t \dot{A}^2(\xi)\mathrm{d}\xi.$$

The energy of transmitted wave motion $E_t(t)$ is

$$E_t(t) = \int_1^{1+\sqrt{H_2}t} \eta_3^2(x,t)\mathrm{d}x = \sqrt{H_2}\int_0^t \dot{B}^2(\xi)\mathrm{d}\xi.$$

Because the dissipation energy is absent in the considered problem, we have then

$$\lim_{t\to\infty}[E_r(t) + E_t(t)] = E_0.$$

DISCUSSION

The beam deflections and wave motions of the fluid for various bottom topographies and the forms of the incident wave have been calculated and will be presented at the Workshop. The local bed elevations cause more prolonged oscillations of the elastic beam and pronounced reflected wave motion.

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REFERENCES

[1] Watanabe, E., Utsunomiya, T. and Wang, C.M., Hydroelastic analysis of pontoon-type VLFS: a literature survey. *Eng. Structures*, 2004, 26, 245-256.

[2] Wang, C.D. and Meylan, M.H., The linear wave response of a floating thin plate on water of variable depth. *Appl. Ocean Res.*, 2002, 24, 163-174.

[3] Sun, H., Cui, W. and Wu Y., Hydroelastic response analysis of VLFS over variable bottom in waves. *Proc. of 3rd Int. Conf. on Hydroelasticity in Marine Technology*, Oxford, UK, 2003, 281-287.

[4] Kyoung, J.H., Hong, S.Y., Kim, B.W. and Cho, S.K., Hydroelastic response of a very large floating structure over a variable bottom topography. *Ocean Eng.*, 2005, 32, 2040-2052.

[5] Meylan, M.H., Spectral solution of time-dependent shallow water hydroelasticity. *J. Fluid Mech.*, 2002, 454, 387-402.

[6] Sturova, I.V., Unsteady behavior of an elastic beam floating on shallow water under external loading. *J. Appl. Mech. Techn. Phys.*, 2002, 43, 415-423.