

Nearfield and farfield boundary-integral representations of free-surface flows

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1. Green's classical potential representation

Consider a finite 3D region \mathcal{D} bounded by a closed surface Σ . The divergence theorem applied to the function $\phi \nabla G - G \nabla \phi$ yields the classical Green identity

$$\int_{\mathcal{D}} d\mathcal{V} (\phi \nabla^2 G - G \nabla^2 \phi) = \int_{\Sigma} d\mathcal{A} (G \mathbf{n} \cdot \nabla \phi - \phi \mathbf{n} \cdot \nabla G) \quad (1)$$

where $d\mathcal{V}$ and $d\mathcal{A}$ stand for differential elements of volume or area of the region \mathcal{D} or the boundary surface Σ , and \mathbf{n} is a unit vector that is normal to Σ and points inside \mathcal{D} . For a function $\phi \equiv \phi(\mathbf{x})$ that satisfies the Laplace equation $\nabla^2 \phi = 0$ within \mathcal{D} , and a Green function $G \equiv G(\mathbf{x}; \tilde{\mathbf{x}})$ that satisfies the Poisson equation $\nabla^2 G = \delta(x - \tilde{x}) \delta(y - \tilde{y}) \delta(z - \tilde{z})$ in \mathcal{D} , or in a larger region that includes \mathcal{D} , (1) yields Green's classical boundary-integral representation

$$\tilde{C} \tilde{\phi} = \int_{\Sigma} d\mathcal{A} (G \mathbf{n} \cdot \nabla \phi - \phi \mathbf{n} \cdot \nabla G) \quad (2a)$$

$$\text{with } \tilde{C} = \int_{\mathcal{D}} d\mathcal{V} \delta(x - \tilde{x}) \delta(y - \tilde{y}) \delta(z - \tilde{z}) = \begin{cases} 1 \\ 1/2 \\ 0 \end{cases} \text{ if } \tilde{\mathbf{x}} \text{ lies } \begin{cases} \text{inside } \mathcal{D} \\ \text{on } \Sigma \\ \text{outside } \mathcal{D} \end{cases} \quad (2b)$$

Here and below, $\mathbf{x} = (x, y, z)$ and ϕ are nondimensional in terms of a reference length L and velocity U , i.e. one has $\mathbf{x} = \mathbf{X}/L$ and $\phi = \Phi/(UL)$. In (2b), the value $\tilde{C} = 1/2$ at a point $\tilde{\mathbf{x}}$ of the boundary surface Σ assumes that Σ is smooth at $\tilde{\mathbf{x}}$. Green's representation (2) defines the potential $\tilde{\phi} \equiv \phi(\tilde{\mathbf{x}})$ at a flow-field point $\tilde{\mathbf{x}}$ in terms of boundary distributions of sources (with strength $\mathbf{n} \cdot \nabla \phi$) and normal dipoles (strength ϕ), and involves a Green function G and the first derivatives of G . In (2) and below, $\tilde{\mathbf{x}}$ stands for a flow-field point, i.e. a point inside a 3D flow region \mathcal{D} , and \mathbf{x} is a point of the boundary surface Σ of the flow region, i.e. a boundary point.

The general solution of the Poisson equation $\nabla^2 G = \delta(x - \tilde{x}) \delta(y - \tilde{y}) \delta(z - \tilde{z})$ is given by

$$4\pi G = -1/r + H = S + H \quad \text{with } r = \sqrt{(x - \tilde{x})^2 + (y - \tilde{y})^2 + (z - \tilde{z})^2} \quad (3)$$

r is the distance between $\mathbf{x} = (x, y, z)$ and $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}, \tilde{z})$, and $H(\mathbf{x}; \tilde{\mathbf{x}})$ stands for a function that is harmonic within the flow region \mathcal{D} (or a larger region that includes \mathcal{D}). Thus, the singular component S and the harmonic component H in (3) satisfy

$$\nabla^2 S = 4\pi \delta(x - \tilde{x}) \delta(y - \tilde{y}) \delta(z - \tilde{z}) \quad \text{and} \quad \nabla^2 H = 0$$

Application of Green's identity (1) to the potential ϕ and the functions S or H yield

$$4\pi \tilde{C} \tilde{\phi} = \int_{\Sigma} d\mathcal{A} (S \mathbf{n} \cdot \nabla \phi - \phi \mathbf{n} \cdot \nabla S) \quad 0 = \int_{\Sigma} d\mathcal{A} (H \mathbf{n} \cdot \nabla \phi - \phi \mathbf{n} \cdot \nabla H) \quad (4)$$

2. Application to free-surface flows in deep water

The boundary surface Σ and the Green function G in Green's relations (2) and (4) are generic. These generic relations are now applied to free-surface flows about ships or offshore structures in deep water. The closed boundary surface Σ in (2a) consists of $\Sigma = \Sigma_B \cup \Sigma_0 \cup \Sigma_\infty$. Here, Σ_B stands for the mean wetted hull of a rigid body (ship or structure) or, more generally, a control surface that encloses a rigid body; Σ_0 is the portion of the mean free-surface plane located outside the "body" surface Σ_B ; and Σ_∞ joins Σ_0 and Σ_D in the farfield. The unit vector $\mathbf{n} = (n^x, n^y, n^z)$ is normal to the boundary surface Σ and points into the flow domain, as already noted. Thus, $\mathbf{n} = (0, 0, -1)$ at the free surface Σ_0 . The Green function G in (2a) is presumed to vanish sufficiently rapidly in the farfield to nullify the contribution of the farfield boundary surface Σ_∞ . Thus, the contribution of Σ_∞ is ignored, and the free surface Σ_0 is unbounded. Green's potential representation (2a) then becomes

$$\tilde{C} \tilde{\phi} = \int_{\Sigma_B} dA (G \mathbf{n} \cdot \nabla \phi - \phi \mathbf{n} \cdot \nabla G) - \int_{\Sigma_0} dx dy (G \phi_z - G_z \phi) \quad (5)$$

Here, the z axis is vertical and points upward, and the mean free surface is taken as the plane $z=0$. It is useful to use a Green function of the form

$$4\pi G = -1/r \pm 1/r_* + H \quad \text{with} \quad r_* = \sqrt{(x - \tilde{x})^2 + (y - \tilde{y})^2 + (z + \tilde{z})^2} \quad (6)$$

$H(\mathbf{x}; \tilde{\mathbf{x}})$ is harmonic within the flow region \mathcal{D} (or a larger region that includes \mathcal{D}). Expressions (2b) and (6) show that $\tilde{C} = 1$ if the flow-field point $\tilde{\mathbf{x}}$ is located at the free surface Σ_0 . Thus, (2b) becomes

$$\tilde{C} = \left\{ \begin{array}{c} 1 \\ 1/2 \\ 0 \end{array} \right\} \text{ if } \tilde{\mathbf{x}} \text{ lies } \left\{ \begin{array}{c} \text{in } \mathcal{D} \cup \Sigma_0 \\ \text{on } \Sigma_B \\ \text{outside } \mathcal{D} \cup \Sigma_B \cup \Sigma_0 \end{array} \right\} \quad (7)$$

3. Local-flow and wave decomposition

The harmonic function H in the basic Green-function representation (6) can be decomposed into a local-flow component and a wave component. Thus, a Green function of the form

$$4\pi G = -1/r + L + W = R + W \quad (8)$$

is now considered. Here, the local-flow component L and the wave component W are presumed to satisfy the Laplace equations $\nabla^2 L = 0$ and $\nabla^2 W = 0$. The decomposition (8) is not unique. For instance, for diffraction-radiation of regular waves by an offshore structure, a particularly simple choice of Green function is defined in [1] as

$$R = -1/r - 1/r_* + 2/r_f \quad W = -i f^2 e^{f^2 Z_*} \int_{-\pi}^{\pi} dt (1 - \Theta) e^{-i \Phi} \quad (9a)$$

$$\text{where } \left. \begin{array}{l} r = \sqrt{h^2 + Z^2} \quad r_* = \sqrt{h^2 + Z_*^2} \quad r_f = \sqrt{h^2 + Z_f^2} \\ h = \sqrt{X^2 + Y^2} \quad X = \tilde{x} - x \quad Y = \tilde{y} - y \\ Z = \tilde{z} - z \quad Z_* = \tilde{z} + z \quad Z_f = Z_* - \sigma^R/f^2 \end{array} \right\} \quad (9b)$$

$$\Phi = f^2 (X \cos t + Y \sin t) \quad \Theta = \frac{\sinh(\Phi/\sigma^W) + i \sin(V/\sigma^W)}{\cosh(\Phi/\sigma^W) + \cos(V/\sigma^W)} \quad \text{with } V = f^2 Z_* \quad (9c)$$

and $-V/\sigma^W < C^W < \pi$. This Green function, which only involves elementary functions of real arguments, satisfies the free-surface condition $G_z - f^2 G = 0$ at $z = 0$ accurately in the farfield, but only approximately (to leading order) in the nearfield. The Rankine component R involves three elementary free-space Rankine sources: a unit source at the singular point $\mathbf{x} = (x, y, z)$, a unit source at the mirror image $(x, y, -z)$ of \mathbf{x} with respect to the mean free-surface plane $z = 0$, and a “double” sink (strength 2) at the point $(x, y, -z + \sigma^R/f^2)$. This arrangement of elementary point sources and sinks satisfies the linear free-surface boundary condition $G_z - f^2 G = 0$ at $z = 0$ to leading order, in both the nearfield and the farfield. The wave component is given by a one-dimensional Fourier superposition of elementary waves $e^{f^2 Z_* - i \Phi}$. The radiation condition is satisfied via the function Θ ; see [1, 2]. More complicated Green functions that satisfy the linear free-surface boundary condition everywhere (in the nearfield as well as the farfield) can be used. These alternative free-surface Green functions are also of the form (8); see e.g. [3, 4].

Using the Rankine-wave decomposition (8) of the Green function in (5), one obtains the representation $4\pi \tilde{C} \tilde{\phi} = \tilde{\phi}^R + \tilde{\phi}^W$ of the potential $\tilde{\phi}$ at a flow-field point $\tilde{\mathbf{x}}$. Here, the Rankine potential $\tilde{\phi}^R$ and the wave potential $\tilde{\phi}^W$ correspond to the Rankine and wave components R and W in (8), and \tilde{C} is given by (7). Thus, for diffraction-radiation of regular waves by an offshore structure, the potentials $\tilde{\phi}^R$ and $\tilde{\phi}^W$ are given by

$$\tilde{\phi}^R = \int_{\Sigma_B} dA (R \mathbf{n} \cdot \nabla \phi - \phi \mathbf{n} \cdot \nabla R) + \int_{\Sigma_0} dx dy [(R_z - f^2 R) \phi - R(\phi_z - f^2 \phi)] \quad (10a)$$

$$\tilde{\phi}^W = \int_{\Sigma_B} dA (W \mathbf{n} \cdot \nabla \phi - \phi \mathbf{n} \cdot \nabla W) + \int_{\Sigma_0} dx dy [(W_z - f^2 W) \phi - W(\phi_z - f^2 \phi)] \quad (10b)$$

Use of expressions (9) for the Rankine and wave components R and W in (10) yields expressions for the potentials $\tilde{\phi}^R$ and $\tilde{\phi}^W$ that only involve elementary functions of real arguments. These simple expressions for $\tilde{\phi}^R$ and $\tilde{\phi}^W$ are given in [5]. Integration over the free surface Σ_0 in (10) only needs to be performed over a finite nearfield portion of the unbounded free surface. In particular, the terms $R_z - f^2 R$ and R in (10a) are $O(1/\rho^3)$ as $\rho \equiv \sqrt{x^2 + y^2} \rightarrow \infty$.

4. A seemingly paradoxical property

The wave component W in (8) satisfies the Laplace equation $\nabla^2 W = 0$, as already noted. It follows from (4) that the wave potential $\tilde{\phi}^W$ in (10) is null, i.e. one has $\tilde{\phi}^W \equiv 0$. However, the wave potential $\tilde{\phi}^W$ is known to become exact in the horizontal farfield; specifically, one has $\tilde{\phi}^W \sim 4\pi\tilde{\phi}$ as $\rho \equiv \sqrt{x^2 + y^2} \rightarrow \infty$. This seemingly contradictory result can be explained if the unbounded free surface Σ_0 is divided into a finite nearfield portion Σ_0^{near} and an unbounded farfield portion Σ_0^{far} . Thus, the unbounded free surface is expressed as $\Sigma_0 = \Sigma_0^{near} \cup \Sigma_0^{far}$. Specifically, Σ_0^{far} is taken here as the region $\rho_\infty < \rho$. Let $\tilde{\phi}_{near}^R$ and $\tilde{\phi}_{near}^W$ stand for the contributions of the finite nearfield boundary surface $\Sigma^{near} = \Sigma_B \cup \Sigma_0^{near}$ to the Rankine and wave potentials $\tilde{\phi}^R$ and $\tilde{\phi}^W$. One has

$$\left. \begin{aligned} \tilde{\phi}_{near}^W &\approx 0 & \text{and} & & \tilde{\phi}_{near}^R &\approx 4\pi\tilde{\phi} & \text{for} & & \rho \leq \rho_{inner} < \rho_\infty \\ \tilde{\phi}_{near}^R &\approx 0 & \text{and} & & \tilde{\phi}_{near}^W &\approx 4\pi\tilde{\phi} & \text{for} & & \rho_\infty < \rho_{outer} \leq \rho \end{aligned} \right\} \quad (11)$$

This property is numerically illustrated and verified here by considering a simple axisymmetric flow generated by a pulsating point source. Specifically, consider the flow defined by the potential

$$\phi(\mathbf{x}) = G(\mathbf{a}; \mathbf{x}) \quad \text{with} \quad \mathbf{a} = (0, 0, -0.1) \quad \text{and} \quad f = 2 \quad (12)$$

Here, G stands for the free-surface Green function defined by (8) and (9) with $\sigma^R = 1$, $\sigma^W = 3$, $C^W = 2.3$. The flow due to the pulsating source (12) is considered for the unbounded flow region that is outside a spherical boundary surface Σ_B , taken as the half sphere $\sqrt{x^2 + y^2 + z^2} = 1$ with $z \leq 0$. The flow $\tilde{\phi}_{near}$ associated with (12) is defined by (10) in terms of the flux $\mathbf{n} \cdot \nabla \phi$ at Σ_B , the potential ϕ at $\Sigma_B \cup \Sigma_0^{near}$ and the pressure $\phi_z - f^2 \phi$ at Σ_0^{near} . These three forcing terms, easily evaluated from (9), are depicted in Fig.1 along the line defined by $y = 0$ and

$$\left. \begin{aligned} x &= \sin(\pi t/2) & z &= -\cos(\pi t/2) & \text{with} & & 0 \leq t \leq 1 \\ x &= t & z &= 0 & \text{with} & & 1 \leq t \leq 18 \end{aligned} \right\} \quad (13)$$

Fig.1 shows that the pressure $\phi_z - f^2 \phi$ at the free surface Σ_0 decays rapidly as t increases, i.e. away from the (spherical) boundary surface Σ_B . The ‘‘input’’ potential $\tilde{\phi}$ given by (12) and the Rankine potential $\tilde{\phi}^R/(4\pi)$ and wave potential $\tilde{\phi}^W/(4\pi)$, reconstructed using the boundary-integral representation (10), are depicted in Fig.2 and Fig.3 along the line

$$\left. \begin{aligned} x &= (1 + \mu) \sin(\pi t/2) & z &= -(1 + \mu) \cos(\pi t/2) & \text{with} & & 0 \leq t \leq t^* \\ x &= t & z &= -\mu & \text{with} & & t^* \leq t \leq 18 \end{aligned} \right\} \quad (14)$$

and $t^* = (2/\pi) \cos^{-1}[\mu/(1 + \mu)]$. The line (14) is located inside the flow region, at a distance μ (taken equal to 0.03 here) from the line (13).

Fig.2 depicts the potential (12) and the Rankine potential $\tilde{\phi}^R/(4\pi)$ with ρ_∞ taken equal to 6 or 12. Thus, the radius ρ_∞ of the nearfield region Σ_0^{near} of the unbounded free surface is taken equal to 6 or 12 in Fig.2. This figure shows that $\tilde{\phi}_{near}^R \approx 4\pi\tilde{\phi}$ in an inner region $\rho \leq \rho_{inner} < \rho_\infty$ and that $\tilde{\phi}^R$ vanishes rapidly for $\rho_\infty < \rho$, in accordance with (11). The radius ρ_{inner} of the inner region, roughly equal to 5 for $\rho_\infty = 6$ or to 11 for $\rho_\infty = 12$, increases as the radius ρ_∞ of the nearfield region Σ_0^{near} of the free surface increases, and one may presume that $\rho_{inner} \rightarrow \infty$ as $\rho_\infty \rightarrow \infty$. Similarly, Fig.3 depicts the potential (12) and the wave potential $\tilde{\phi}^W/(4\pi)$ with ρ_∞ taken equal to 6 or 12, as in Fig.1. Fig.2 shows that $\tilde{\phi}_{near}^W \approx 4\pi\tilde{\phi}$ in an outer region $\rho_\infty < \rho_{outer} \leq \rho$ and that $\tilde{\phi}_{near}^W \approx 0$ in an inner region $\rho \leq \rho_{inner} < \rho_\infty$, in agreement with (11). Fig.3 shows that one has $\rho_{inner} \approx 1$ and $\rho_{outer} \approx 10$ for $\rho_\infty = 6$; and $\rho_{inner} \approx 6$ and $\rho_{outer} \approx 16$ for $\rho_\infty = 12$. Thus, the radius ρ_{inner} of the inner region increases with the size of the nearfield free-surface region Σ_0^{near} , and one may presume that $\rho_{inner} \rightarrow \infty$ as $\rho_\infty \rightarrow \infty$.

5. Additional results

Implications of the property (11) and related numerical results given in Fig.2 and Fig.3 will be discussed at the Workshop. Alternative boundary integral representations suited for nearfield and farfield flows will also be presented, and numerically illustrated for wave diffraction-radiation by an offshore structure. A more detailed account of this study will be reported in [5].

References

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Figures

Fig.1 (top of right column) Flux $\mathbf{n} \cdot \nabla \phi$ at Σ_B , potential ϕ at $\Sigma_B \cup \Sigma_0^{near}$ and pressure $\phi_z - f^2 \phi$ at Σ_0^{near} . The free-surface pressure $\phi_z - f^2 \phi$ vanishes rapidly in the farfield.

Fig.2 (center of right column) Input potential $\tilde{\phi}$ given by (12) and Rankine potential $\tilde{\phi}^R/(4\pi)$ for free-surface integration truncated at $\rho_\infty = 6$ and $\rho_\infty = 12$. The Rankine potential $\tilde{\phi}^R$ is exact in the nearfield and vanishes rapidly in the farfield.

Fig.3 (bottom of right column) Input potential $\tilde{\phi}$ given by (12) and wave potential $\tilde{\phi}^W/(4\pi)$ for free-surface integration truncated at $\rho_\infty = 6$ and $\rho_\infty = 12$. The wave potential $\tilde{\phi}^W$ is null in the nearfield and becomes exact in the farfield.

