

Structural Acoustics of a Floating Circular Elastic Plate

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Reducing the radiated noise induced by elastic structures in a fluid (water or air) is a practical problem of immense importance. It is also well known that passive means of decreasing the noise level, especially at low frequencies are generally not very effective. The possibility of using active noise control which are applied directly on the structure has been recently suggested, triggered by the advent of smart structures and the advances in fast micro-processors combined with modern piezoelectric actuators or other induced strain transducers (e.g. Fuller et al 1997). Our interest resides mainly in the technological aspects of fluid-loading of a submerged vibrating structure in the range of frequencies where the compressibility of the fluid medium should be taken into account. Examples for the significance of the interaction between sound and vibration abound in marine engineering (ships, submarine, oil rigs submerged pipes, etc.) as well as in aeronautical, mechanical and nuclear engineering (e.g. Crighton 1989). For the sake of simplicity, these structures are considered here as planar elastic surfaces of finite extent lying on elastic foundations modeled by an otherwise quiescent compressible fluid. The simplest 3-D geometry that one can envisage is that of an isotropic elastic finite buoyant circular plate lying on the interface between two immiscible fluids (including air-water) which is forced harmonically to deform. The motion and deformation of the elastic plate are coupled with those induced in the compressible fluid.

In spite of the fact that we employ linear models to describe the elastic plate deflections (bi-harmonic Kirchhoff-Euler) and using the acoustic approximation for the fluid, the analysis is far from being trivial and a closed-form solution can not be obtained. A somewhat simplified problem has been first posed by Rayleigh (1896) in his seminal paper "Reaction of the air on a vibrating circular plate". However, this solution is limited in the sense that it failed to consider the full fluid-structure coupling which is manifested by the significant fluid loading effect between the structural vibration and fluid motion. Such coupling appears also in modeling the interaction action between elastic structures and supporting soil media (Wang et al 2005), and it is discussed in the recent text of Szilard (2004).

Let us consider next a circular plate of radius a and of thickness h lying on a free-surface (water-air) $z = 0$, where z is pointing downward in the direction of gravity. The plate is excited harmonically by some prescribed forcing system (discrete or continuous) $F(r, \varphi)e^{-i\omega t}$, where F is the forcing complex amplitude, ω is the forcing frequency and (r, φ) are polar coordinates in the $z = 0$ plane. In a similar manner, the elastic (complex) deflection of the plate relative to the ambient interface is represented by $W(r, \varphi)e^{-i\omega t}$, where it is understood that only the real part of such a product should be considered. For the practical range of parameters (depending on excitation frequency and plate radius), the fluid must be treated as a compressible medium, where the complex pressure $P(r, \varphi, z)e^{-i\omega t}$ under the acoustical approximation (e.g. Lighthill 1978, Ch.1.) is governed by the Helmholtz wave equation:

$$(\nabla^2 + \tilde{k}^2)P = 0 \quad (1)$$

where $\tilde{k} = \omega/c$ denotes the acoustic wave number of the medium and c represents the sound velocity in the fluid. The pressure term is also subject to a proper radiation (decay) condition at infinity. Expressed in the local planar coordinate system the complex plate deflection (after suppressing the $e^{-i\omega t}$ dependence) can be next written by using the linearized Reissner plate model (Szilard 2004) and including an acoustic linear damping term (e.g. Bermudez et al 2001, Mellow & Karkainen 2006).

$$(D\nabla^4 - i\omega z_s - m\omega^2)W(r, \varphi) = P(r, \varphi, 0) + F(r, \varphi) - \frac{h^2}{10} \left(\frac{2-\nu}{1-\nu} \right) \nabla^2 |P(r, \varphi, 0) + F(r, \varphi)| \quad 0 < r < a. \quad (2)$$

The last term includes a correction due to the finite plate thickness (h) in terms of the surface Laplacian of the forcing where ν is the plate Poisson's ratio. For thin plates this extra term is generally ignored, alternatively higher non-linear models, of various Mindlin's type can be also used (e.g. Alper & Magrad 1970, Bermudez et al 2001, etc.). The damping term of impedance z_s (assumed to be frequency independent) is introduced here to model a thin layer of viscoelastic coating material between the structure and the fluid with a purpose to

damp the viscoelastic acoustic energy. The plate rigidity $D = \frac{Eh^3}{12(1-\nu^2)}$ is expressed in terms of Young's modulus E and m represents the mass per unit area of the plate.

It is noted that the plate deformations are only restricted to the vertical direction. If in addition to elastic deformation, the plate also moves in a rigid-body heave motion with a constant complex amplitude $W e^{-i\omega t}$, then the Euler equation (relating fluid acceleration and pressure gradient), when applied on the lower (water) side of the plate yields,

$$P_z(r, \varphi, 0) = \rho \omega^2 (\bar{W} + W(r, \varphi)), \quad a > r > 0 \quad (3)$$

where ρ is the density of the fluid. Thus, the normal fluid pressure gradient on the plate is proportional to the plate deformation (typical for elastic foundations), representing the full fluid-structure coupling.

It is remarked that (2) is applied on the plate region $a > r > 0$ and consequently a proper boundary condition should be enforced on the rest of the $z = 0$ plane ($r > a$). It will be demonstrated in the sequel that in the range of parameters of interest the free-surface can be considered as a pressure-free (soft) surface, where

$$P(r, \varphi, 0) = 0 \quad r > a \quad (4)$$

To complete the formulation for the hydroelastic plate response some boundary conditions must be satisfied on the edge of the plate $r = a$. Typically such boundary conditions can be either; a) clamped end (i.e., $W(a, \varphi) = \nabla W(a, \varphi) = 0$); b) simply supported end (i.e., $W(a, \varphi) = M(a, \varphi) = 0$) and c) free end

$M(a, \varphi) = Q(a, \varphi) = 0$ where $M(a, \varphi)$ is the bending moment (Zilman & Miloh 2000)

$$M(r, \varphi) = -D \left[\nabla^2 - \frac{(1-\nu)}{r} \left(\frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial^2}{\partial \varphi^2} \right) \right] W(r, \varphi) \quad (5)$$

and $Q(r, \varphi)$ is the vertical shear force

$$Q(r, \varphi) = -D \left[\frac{\partial}{\partial r} \nabla^2 + \frac{(1-\nu)}{r} \left(\frac{\partial}{\partial r} - \frac{1}{r} \right) \frac{\partial^2}{\partial \varphi^2} \right] W(r, \varphi) \quad (6)$$

The model problem is thus a coupled and mixed b.v.p involving a fourth-order p.d.e. Using dimensional representation in which we employ a and ω^{-1} as the characteristic length and times scales respectively, the governing equations for a thin elastic plate are summarized below:

$$(\nabla^2 + k^2) P = 0 \quad \text{in } \forall_f \quad (7)$$

$$(\alpha \nabla^4 - \beta) P_z - P = F \quad \text{on } S_p \quad (8)$$

$$P_z = \bar{W} + W \quad \text{on } S_p \quad (9)$$

$$P = 0 \quad \text{on } S_{FS} \quad (10)$$

and proper edge conditions to be applied on the contour of the plate ($S_p \cup S_{FS}$) expressed in terms of the prescribed differential operators

$$\mathcal{L}_1(P_z) = \mathcal{L}_2(P_z) = 0 \quad (11)$$

Here $k = \tilde{k}a$, $\alpha = \frac{D}{\rho \omega^2 a^5}$ and $\beta = \frac{m}{\rho a} \left(1 + i \frac{z_s}{m \omega} \right)$.

Let the structural wave number in vacuo be given by $k_p = (m \omega^2 / D)^{1/4}$ and the ‘acoustic’ Mach number be defined by the ratio $M = k / k_p$, where k is the acoustic wave number. One can then specify (Crighton 1989) for $M = 1$ a ‘coincidence’ frequency $\omega_g = (m c^4 / D)^{1/2}$. In general, forcing frequencies where $\omega > \omega_g$ define a ‘supersonic’ case and $\omega < \omega_g$ represents a ‘subsonic’ regime. It is also noted that in the subsonic case, the structural wave is essentially unchanged by the fluid loading compared to vacuum. In most engineering applications, supersonic conditions prevail and compressibility effects should be accounted for.

The present set of equations, although linearized, do not yield a closed-form analytic solution and are usually solved by employing approximate methods (e.g. expansion in ‘dry’ modes, mixed integral transforms, variational methods based on BEM etc.). One of such methods is based on using eigenfunction expansions in which we express the solution independently in the two regions (i.e., $r < 1$ and $r > 1$) and match the two expansions on the fictitious cylindrical surface $r = 1$. Examples can be found in (Zilman & Miloh 2000 and Sturova 2002) which consider wave excitation of a circular plate in the incompressible (non-acoustic) case. Difficulties resulting from such matching, even in shallow water, are discussed in (Zilman & Miloh 2000).

In order to obtain a unified solution which holds everywhere in $\forall_f$, we use oblate spheroidal coordinates (μ, ζ, φ) which are related to the cylindrical (r, φ, z) set by

$$z = \mu\zeta \quad ; \quad r^2 = (1 - \mu^2)(\zeta^2 + 1) \quad ; \quad 1 > \mu > 0 \quad , \quad \zeta > 0 \quad (12)$$

The advantage of such separable representation is that S_p is given by $\zeta = 0$ where S_{FS} by $\mu = 0$. The acoustic wave equation (7) yields a formal solution in terms of the so-called ‘spheroidal wave functions’ (see for example Flammer 1957) which can be expressed in the axisymmetric case as

$$P(\mu, \zeta) = \sum_{n=0}^{\infty} A_{2n+1}(k) U_{2n+1}(\mu) V_{2n+1}(\zeta) \quad (13)$$

where the A 's are coefficients to be determined and U and V can be defined in terms of the appropriate spheroidal wave functions $S^{(1)}$ and $R^{(4)}$ (using Flammer's notation) as

$$U_{2n+1}(\mu) = \frac{S_{2n+1}^{(1)}(-ika, \mu)}{S_{2n+1}^{(1)}(-ika, 1)} \quad ; \quad V_{2n+1}(\zeta) = \frac{R_{2n+1}^{(4)}(-ika, \zeta)}{R_{2n+1}^{(4)}(-ika, 0)} \quad (14)$$

where the upper dot denotes differentiation with respect to the argument (ζ in this case). Furthermore, the fact that the sum in (13) includes only odd values of n , renders automatically that $P = 0$ on $S_{FS}(\mu = 0)$ in accordance with (10). It is noteworthy that a similar formulation in terms of spheroidal wave functions can be used in order to determine the acoustic radiation from spheroidal-like submerged bodies (e.g. Boisvert & van Buren 2004, Rapids & Lauchle 2006).

The normal pressure gradient on the plate, expressed in the same spheroidal coordinate system (12) is $\frac{\partial p}{\partial n} = \frac{1}{\mu} \frac{\partial p}{\partial \zeta} \Big|_{\zeta=0}$ and thus eqs. (9), (13) and (14) imply that

$$W(r) = P_z(r, 0) = \sum_{n=0}^{\infty} A_{2n+1} \frac{U_{2n+1}(\mu)}{\mu} \quad ; \quad \mu = \sqrt{1 - r^2} \quad (15)$$

A practical way of computing the eigenfunctions $U_m(\mu)$ is given for example in Aoi (1955), Flammers (1957) and Bouwkamp (1970). In particular it is shown that for small and moderate values of the dimensionless acoustic wave number k , one can expand $U_m(\mu)$ in terms of the Legendre polynomials $P_m(\mu)$ as

$$U_n(\mu) = \sum_{m=0}^{\infty} b_n^{(m)}(k) P_m(\mu) \quad (16)$$

where $b_n^{(m)}$ are prescribed coefficients depending on k . Satisfying the two edge conditions on the contour of the disk (11) imply that

$$\sum_n \sum_m A_{2n+1} b_n^{(m)} \mathcal{L}_{1,2} \left(\frac{P_{2m+1}(\mu)}{\mu} \right) \Big|_{\mu=0} = 0 \quad (17)$$

Thus, determining for example the values of the first two coefficients (A_1, A_3) in terms of the rest. The complete solution for the unknown coefficients (functions of k) is then found by substituting (14) and (15) into the plate mixed-boundary condition (18) applied on $\zeta = 0$, resulting in

$$\sum_{n=0}^{\infty} A_{2n+1} \left[(\alpha \nabla^4 - \beta) \left(\frac{U_{2n+1}(\mu)}{\mu} \right) - V_{2n+1}(0) U_{2n+1}(\mu) \right] = F(\mu) \quad (18)$$

where $V_{2n+1}(0)$ is a constant depending on k and $F(\mu)$ is the prescribed forcing on the upper plate which can generally be expressed in terms of the complete set $U_m(\mu)$ as

$$F(\mu) = \sum_{n=0}^{\infty} \gamma_n U_n(\mu) \quad (19)$$

Such a representation is convenient in view of the fact that the eigenfunctions $U_m(\mu)$ are orthonormal with (Aoi 1955, Flammer 1957, Bouwkamp 1970)

$$\int_{-1}^1 U_m(\mu) U_n(\mu) d\mu = \frac{2}{2n+1} \delta_{mn} \quad (20)$$

Also note that U_{2n} and U_{2n+1} are even and odd functions of μ respectively. Thus, multiplying (18) by $U_m(\mu)$ and integrating over $[0,1]$ leads to a set of linear equations which in principal uniquely determine the coefficients A_{2n+1} to any order of accuracy.

Once the coefficients are found, it is then straightforward to compute the sought parameters. For example, for a central forcing the maximum plate deflection at the center is given by (15)

$$W_{\max} = W(0) = \sum_{n=0}^{\infty} A_{2n+1} \quad (21)$$

The far-field pressure field induced by the plate forcing can be found directly from (13) by recalling that at a large distance R from the origin (see Bouwkemp 1970) one has

$$\lim_{R \rightarrow \infty} V_{2n+1}(\zeta) = C_{2n+1} \frac{e^{-ikR}}{R} \quad (22)$$

where the coefficients $C_{2n+1}(k)$ are again some given functions of k . Thus the far-field pressure is given by

$$P(\theta) = f(\theta) \frac{e^{-ikR}}{R} ; f(\theta) = \sum_{n=0}^{\infty} C_{2n+1} A_{2n+1} U_{2n+1}(\theta) \quad (23)$$

where $f(\theta)$ represents the directivity function (see Pierce 1991, p. 225) and $\mu = \cos \theta$. A key parameter in physical acoustics is the so-called far-field intensity representing the amount of acoustical energy radiated away from the source. This quadratic function also serves as the cost function to be minimized for a global control of sound radiation from the plate under a prescribed loading (Fuller et al. 1997, p. 246) and is generally given in terms of the far -field pressure and the transmission coefficient T as,

$$J = \frac{1}{2\rho c} \int_{S_p} P P^* dS = \frac{T}{2\rho c R^2} \quad (24)$$

where S_p denotes the plate area and the upper star denotes complex conjugate. For the present circular plate and by using (20) and (23), the total acoustic transmission coefficient can be expressed as

$$T = \sum_{n=0}^{\infty} \frac{2}{4n+3} (C_{2n+1} A_{2n+1})^2 \quad (25)$$

Sample calculations of $T(k)$ and plate deflections for a clamped disk, excited by a plane wave and controlled

by a central point loading, $\left(F(r) = \frac{\delta(r)}{2\pi r} \right)$, for supersonic conditions, will be presented at the workshop.

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