Nonlinear wave-body interaction by a formulation in spectral space

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Introduction

The need for prediction tools for strongly nonlinear wave-body interaction poses new challenges within the field of marine hydrodynamics. This is an area that is very open. We foresee requests from industry, for example, for a next generation of wave analysis tools that are beyond the second and third order theories that routinely are used today. Several special solutions exist, for geometries of special shape and in cases where the flow everywhere can be assumed to be smooth. Numerical solutions are derived for local splashy and breaking flows, but the procedures are not easily adaptive to the complete wave diffraction-radiation problem, however.

Here we are testing out a formulation in wavenumber space where a part of the solution locally may contain breaking. More generally, we are deriving an outer solution that is smooth such that potential theory can be used, and is obtained in Fourier space. An inner solution may be non-smooth and contain local breaking. It may be computed by a numerical method that can tackle fluid motion that breaks locally or violently runs up along a geometry including green water on deck. The motion is highly nonlinear which means a significant transfer of energy to high wavenumbers. This also means that the effect is local in physical space.

A domain decomposition method in two dimensions has recently been tested out. The less violent flow at a distance from the ship is described using a boundary element method. This reduces the computational time which is a major limiting factor for practical applications. Two-dimensional results reported so far are promising, see e.g. Greco, Colicchio and Faltinsen (2005). A strategy for real world problems should involve the computations of realistic geometries in three dimensions, and represents the next step.

In this presentation we shall be concerned with the outer flow problem. The derivations are put into the perspective of a method that is under development and testing. It is interesting to test how well the method will perform in practice, and if it is useful or not. Here we will describe how to obtain the smooth flow outside a fixed vertical circular cylinder contour that is exposed to incoming nonlinear waves. The waves are nonbreaking and have finite vertical excursion along the cylinder contour. Variants of this problem has previously been studied by several authors, and solutions are obtained in different ways. A recent method was developed by Bonnefoy et al. (2006). Here, we obtain the mathematical solution in spectral space. Eventual matching strategies will follow the developments in two dimensions and will not be discussed here. A preliminary version of the equations was presented in Grue (2005). Here we take next step and solve the integral equation on the body surface that comes out of the formulation.

Equations for the motion of the free surface

The formulation assumes the application of potential theory. The wave potential is denoted by ϕ. We introduce horizontal space coordinates by (x_1, x_2), vertical coordinate by y and let t denote time. Further, we let $\widetilde{\phi}$ denote the value of the potential at the free surface. In order to integrate the wave surface forward in time we need to know the normal velocity of the wave surface obtained by $V = \partial\phi/\partial n\sqrt{1+|\nabla\eta|^2}$.

Let $\mathcal{F}$ denote Fourier transform over the free surface where the values inside the cylinder are put to zero. The variable $\mathcal{F}(V)$ is obtained by an iteration where the leading contributions are (Grue, 2005)

$$\frac{\mathcal{F}(V_1)}{k} = \mathcal{F}(\widetilde{\phi}) + \Phi_{B1}, \tag{1}$$

$$\frac{\mathcal{F}(V_2)}{k} = -\mathcal{F}(\eta V_1) - \mathbf{i}\frac{\mathbf{k}}{k}\cdot\mathcal{F}(\eta\nabla\widetilde{\phi}), \tag{2}$$

where $V = V_1 + V_2 + ...$, η denotes the wave elevation outside the cylinder, ∇ the horizontal gradient, $\mathbf{k}$ the wavenumber and $k = |\mathbf{k}|$. Equations (1)–(2) represent the first steps of an iteration to obtain the fully nonlinear wave field (expressed by V, η and $\widetilde{\phi}$). This iteration strategy is very accurate and efficient and has been tested out in the case when there is no geometry in the fluid ($\Phi_{B1} = 0$), see Fructus et al. (2005).

The new task is to evaluate the contribution due to the geometry which is expressed by the integral

$$\Phi_{B1}(\mathbf{k}) = -\frac{\partial}{\partial K}\int_0^{2\pi}\int_{-\infty}^{0}\psi'_B(z',\theta')e^{-\mathbf{i}K\cos(\alpha-\theta')}dz'ad\theta', \tag{3}$$

where a denotes the cylinder radius, $K = ka$ and $\mathbf{k} = k(\cos\alpha, \sin\alpha)$. In (3) we have replaced the wave potential at the body surface, $\phi_B(y,\theta)$, by $\psi_B(z,\theta)$, where y and z are related by $y = z + \eta_0(\theta)$, and $\eta_0(\theta)$ denotes the wave elevation along the water line of the cylinder. The integral in (3) requires the solution of the wave potential ψ_B at the cylinder surface. (There are also contributions by some higher order moments of the integral.)

Integral equation for the wave potential at the geometry

Let

$$\mathcal{L}_0(\phi_B) = \phi_B + \Big(\frac{1}{2\pi}\int_{S_B}\phi'_B\frac{1}{r}\Big)_0 + \mathcal{F}^{-1}\{e^{kz}[-\Phi_{B1}]\}, \tag{4}$$

$$H_0 = 2\mathcal{F}^{-1}\{e^{kz}\mathcal{F}(\widetilde{\phi})\}, \tag{5}$$

where r denotes the distance between two points on the body surface, and z is related to y by $y = z + \eta_0$. The integral equation reads

$$\mathcal{L}_0(\phi_B) = H_0(\widetilde{\phi}) + \text{remainder} \tag{6}$$

where the remainder is obtained by an implicit formulation including all nonlinear terms. The task is to invert $\mathcal{L}_0(\phi_B)$. Let

$$\psi_B(z,\theta) = \sum_{m=-\infty}^{\infty} f_m(z)e^{\mathbf{i}m\theta}, \quad f_{-m}(z) = f_m^*(z). \tag{7}$$

Then (1)–(2) gives

$$\begin{aligned}f_m(z) - a\int_{-\infty}^{0} dz' f_m(z') \int_0^\infty kdk J_m(ak) J_m'(ak) e^{-k|z-z'|} \\ +a\int_0^\infty kdke^{kz} J_m(ak) J_m'(ak)\int_{-\infty}^{0} dz' f_m(z') e^{kz'} dz' \\ = \frac{2\mathrm{i}^m}{(2\pi)^2}\int_0^\infty kdke^{kz} J_m(ak)\int_0^{2\pi} d\alpha e^{-\mathrm{i}m\alpha}\mathcal{F}(\widetilde{\phi}).\end{aligned} \tag{8}$$

Let $e^z = (1+u)/2$. Then $(-\infty, 0] \to [-1, 1]$. Let $f_m(z) = F_m(u)$. The equation becomes

$$\begin{aligned}F_m(u) + \int_0^\infty kdk J_m(ak) J_m'(ak) K_m(u,k) \\ = \frac{1}{\pi}\int_0^\infty kdk\Big(\frac{1+u}{2}\Big)^k J_m(ak)\int\int_{-\infty}^{\infty} J_m(kr')\widetilde{\phi}(\mathbf{x}')d\mathbf{x}',\end{aligned} \tag{9}$$

where J_m denotes Bessel function of the first kind of order m. The kernel is expressed by

$$K_m(u,k) = -a\int_{-\infty}^{0} dz' f_m(z') e^{-k|z'-z|} + ae^{kz}\int_{-\infty}^{0} dz' f_m(z') e^{kz'}. \tag{10}$$

Let $F_m(u) = \sum_{n=0}^{\infty} B_n^m T_n(u)$ where $T_m(u)$ are the Chebychev polynomials on the interval $[-1, 1]$. This gives

$$\begin{aligned}\Delta_{in} B_i^m \; +a\sum_{n=0}^{\infty} B_n^m \int_0^\infty dk J_m(ak) J_m'(ak)[-\pi\delta_{in} + \mathcal{T}_{in}(k) + kg_i(k)h_n(k)] \\ = \frac{1}{\pi}\int_0^\infty kdkg_i(k) J_m(ak)\int\int_{-\infty}^{\infty} J_m(kr')\widetilde{\phi}(\mathbf{x}')d\mathbf{x}'\end{aligned} \tag{11}$$

to determine the coefficients B_n^m, where,

$$h_n(k) = \frac{1}{k} - \frac{n}{k}\int_0^\pi \sin(ns)\Big(\frac{1+\cos s}{2}\Big)^k ds, \tag{12}$$

$$g_i(k) = \int_0^\pi \cos is\Big(\frac{1+\cos s}{2}\Big)^k ds, \tag{13}$$

$$\mathcal{T}_{in}(k) = \int_0^\pi \cos is\, ds \int_0^\pi n\sin ns'\, ds' f_2(s,s'), \tag{14}$$

$$f_2(s,s') = \left(\frac{1+\cos s'}{1+\cos s}\right)^k, \quad \frac{1+\cos s'}{1+\cos s} \le 1, \tag{15}$$

$$f_2(s,s') = -\left(\frac{1+\cos s}{1+\cos s'}\right)^k, \quad \frac{1+\cos s}{1+\cos s'} < 1, \tag{16}$$

and $\Delta_{00} = \pi$, $\Delta_{in} = \delta_{in}\pi/2$, $n \ge 1$. We end up by evaluating integrals of the kind

$$\int_0^\infty J_m^2(k)\Big(\frac{1+C'}{1+C}\Big)^k dk = \int_0^\infty J_m^2(k) e^{-\beta k} dk = \frac{1}{\pi} Q_{m-\frac{1}{2}}(1 + \tfrac{1}{2}\beta^2), \tag{17}$$

$$\beta = \Big|\ln\frac{1+C'}{1+C}\Big|, \quad C' = \cos s', \quad C = \cos s, \tag{18}$$

where $Q_\nu(Z)$ are Legendre functions of the second kind. The functions are obtained by means of a recursion, i.e.

$$Q_{\nu+1}(Z) = \frac{1}{\nu+1}\Big[(2\nu+1)ZQ_\nu(Z) - \nu Q_{\nu-1}(Z)\Big] \tag{19}$$

etc. The two first contributions are evaluated by

$$Q_{-\frac{1}{2}}(\zeta) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{2^{\frac{1}{2}}\Gamma(1)}\zeta^{-\frac{1}{2}}F(\tfrac{3}{4},\tfrac{1}{4};\tfrac{5}{4};\zeta^{-2}), \tag{20}$$

$$Q_{\frac{1}{2}}(\zeta) = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{2^{\frac{3}{2}}\Gamma(2)}\zeta^{-\frac{3}{2}}F(\tfrac{5}{4},\tfrac{3}{4};\tfrac{7}{4};\zeta^{-2}), \tag{21}$$

where

$$\frac{1}{\zeta^2} = \frac{1}{\Big(1+\frac{1}{2}\Big|\ln\frac{1+C'}{1+C}\Big|^2\Big)^2}, \tag{22}$$

and F denotes the Hypergeometric function defined by

$$F(\tfrac{3}{4},\tfrac{1}{4};\tfrac{5}{4};\zeta^{-2}) = \tfrac{1}{B(\frac{1}{4},\frac{5}{4})}\textstyle\int_0^1 t^{-\frac{3}{4}}(1-t/\zeta^2)^{-\frac{3}{4}}dt, \tag{23}$$

$$F(\tfrac{5}{4},\tfrac{3}{4};\tfrac{7}{4};\zeta^{-2}) = \tfrac{1}{B(\frac{3}{4},\frac{7}{4})}\textstyle\int_0^1 t^{-\frac{1}{4}}(1-t/\zeta^2)^{-\frac{5}{4}}dt, \tag{24}$$

where $B(\frac{1}{4},\frac{5}{4}) = \Gamma(\frac{5}{4})/[\Gamma(\frac{1}{4})\Gamma(1)]$ and $B(\frac{3}{4},\frac{7}{4}) = \Gamma(\frac{7}{4})/[\Gamma(\frac{3}{4})\Gamma(1)]$.

The formulation is under implementation. Numerical results for linear and second order waves will be displayed at the Workshop. Although the scheme outlined in the present version appears in a perturbation like form, it is fully nonlinear. It is just to continue the iteration taking into account the full contributions on the r.h.s. of the integral equations. It is important to note that, if the leading terms of the iteration are easily obtained, then the iteration will be efficient. Once the body potential is obtained, the function (3) is evaluated, and the entire wave field can be integrated forward in time.

References

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