The Scattering by a Sea Ice/Ice Shelf Transition

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1 Introduction

A method is outlined for finding the scattering of ice-coupled flexural-gravity waves by a region of variable thickness connecting two semi-infinite ice sheets of different thickness. With appropriately chosen values for the two outer thicknesses, a linear increase in thickness from the thinner sheet to the larger one, this can be used to model a sea ice/ice shelf transition such as the one that occurs in the Ross Sea.

A full description of the solution method is given by Williams (2005), who also presents various results that were obtained by using it. Chung & Fox (2002) solved a similar problem, that of an abrupt jump in ice thickness, without a transition region, using the Wiener-Hopf technique. Their Wiener-Hopf equations can also be derived by using Green's theorem and taking the Fourier transform of the resulting integral equation. This approach can be extended to allow for the variable region, resulting in a similar Wiener-Hopf type integral equation, but one that is also coupled with a second integral equation over the region of variable thickness. The latter is solved numerically using the method of Williams & Squire (2004).

2 Equations and Boundary Conditions

Figure 1 illustrates the physical situation that we are modelling. A plane wave with unit amplitude and radial frequency ω arrives at a region of variable thickness from beneath the left-hand ice sheet and is partial reflected by and partially transmitted through the variable region into the water beneath the right-hand ice sheet. The amplitudes of the reflected and transmitted waves are R and T respectively. R and T shall be called the reflection and transmission coefficients, and their determination is the main purpose of our solution.

The ice in the left-hand region has thickness h_0, the central region has variable thickness $h_1(x)$, and the right-hand region has thickness h_2. Subscripts of $j = 0$, 1 or 2 will be used to denote quantities referring to the different areas in the same way that they are in the h_j.

For the ice in each region, let us now define the flexural rigidity, $\overline{D}_j = E_j h_j^3/12(1-\nu_j^2)$, and the mass per unit area, $\overline{m}_j = \rho_j h_j$, where E_j, ν_j and ρ_j are the Young's modulus, Poisson's ratio and density of the plate in the j^{th} region (respectively), while ρ is the water density and g is the acceleration due to gravity. In the following we will assume that $h_2 \geq h_1(x) \geq h_0$, and we will define the natural length that we will nondimensionalise with respect to as $L = (\overline{D}_2/\rho\omega^2)^{1/5}$. We will also assume that $E_j = E$, $\rho_j = \rho_{\text{ice}}$ and $\nu_j = \nu$, i.e., the main variation in ice properties is in the thickness.

If we asume that the sea water beneath the ice is inviscid and of constant density, and that the fluid flow is irrotational, then there exists a potential function $\Phi(\bar{x}, \bar{y}, \bar{z}, t)$ such that the velocity of a fluid particle is given by the gradient of Φ. $(\bar{x}, \bar{y}, \bar{z}) = L(x, y, z)$, where x, y and z are nondimensional. Since the forcing from the incident wave is periodic in time, and since the geometry of the problem is shift-invariant in the y direction, we assume that Φ has the following

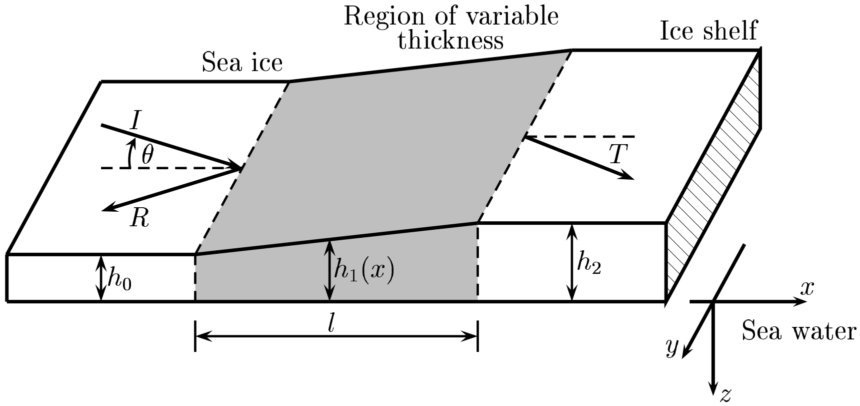

Figure 1: The physical situation to be modelled. A plane flexural-gravity wave arrives from beneath the sea ice and is partially reflected by and partially transmitted through the variable region into the water beneath the ice shelf. The width of the variable regions is l, and the thicknesses of the ice in the regions are denoted h_j $(j = 0, 1, 2)$. The ice is modelled using the Euler-Bernoulli thin plate model. Submergence is neglected, so the bottom of each plate is taken to be in the $z = 0$ plane. The left-hand edge of the variable region is located in the $x = 0$ plane (the coordinate axes are displaced to the right to avoid clutter), the incident waves arrives at an angle θ from normal incidence, and the sea water has a finite depth of H.

form:

$$\Phi(\bar{x}, \bar{y}, \bar{z}, t) = \mathrm{Re}\left[\phi(x, z) \times \frac{\mathrm{e}^{\mathrm{i}(\alpha_y y - \omega t)}}{L^2\omega}\right]. \tag{1}$$

This reduces the dimension of the problem from four to two. The (nondimensional) wavenumber α_y will be related to the incoming wave's angle of incidence.

The other two significant lengths, l and H, are also nondimensional, having also been scaled by L, so the dimensional ramp width and water depth are given by $l \times L$ and $H \times L$. Further quantities that we will refer to are

$$D_j = \overline{D}_j / \overline{D}_2, \quad m_j = \overline{m}_j / \overline{m}_2, \quad \lambda = \frac{g}{L\omega^2} - \mathrm{i}\varepsilon, \quad \mu = \frac{m_2}{\rho L},$$

where ε is an infinitesimal quantity introduced to force the reflected and transmitted waves to decay exponentially as they travel away from the central ice strip. The limit as it becomes zero will be taken once the solution has been completed.

$\phi(x, z)$ must satisfy the following system of equations

$$\left(\nabla^2 - \alpha_y^2\right)\phi(x, z) = 0, \tag{2a}$$

$$\boldsymbol{\mathcal{L}}(x, \partial_x)\phi_z(x, 0) + \phi(x, 0) = 0, \tag{2b}$$

$$\phi_x(x^+, z) - \phi_x(x^-, z) = \phi(x^+, z) - \phi(x^-, z) = 0, \tag{2c}$$

$$\phi_z(x, H) = 0, \tag{2d}$$

where

$$\boldsymbol{\mathcal{L}}(x, \partial_x) = (\partial_x^2 - \alpha_y^2) D(x) (\partial_x^2 - \alpha_y^2) + (1 - \nu)\alpha_y^2 + \lambda - m(x)\mu.$$

The function $D(x)$ is defined as taking the value of $D_j(x)$ in the j^{th} region, and $m(x)$ is defined analogously in terms of the $m_j(x)$.

As well as applying the above equations and the radiation conditions inherent in the problem, the full solution must also satisfy some conditions at the two edges $x_e = 0$ and $x_e = l$. The conditions that are most applicable in the modelled situation are as follows:

$$\phi_z(x_e^+,0) = \phi_z(x_e^-,0), \tag{3a}$$
$$\phi_{zx}(x_e^+,0) = \phi_{zx}(x_e^-,0), \tag{3b}$$
$$\boldsymbol{\mathcal{M}}(x_e^+,\partial_x)\phi_z(x_e^+,0) = \boldsymbol{\mathcal{M}}(x_e^-,\partial_x)\phi_z(x_e^-,0), \tag{3c}$$
$$\boldsymbol{\mathcal{S}}(x_e^+,\partial_x)\phi_z(x_e^+,0) = \boldsymbol{\mathcal{S}}(x_e^-,\partial_x)\phi_z(x_e^-,0), \tag{3d}$$

where if $\boldsymbol{\mathcal{L}}_\pm(\partial_x) = (\partial_x^2 - \alpha_y^2) \mp (1-\nu)\alpha_y^2$,

$$\boldsymbol{\mathcal{M}}(x,\partial_x) = D(x)\boldsymbol{\mathcal{L}}_-(\partial_x), \quad \boldsymbol{\mathcal{S}}(x,\partial_x) = D(x)\boldsymbol{\mathcal{L}}_+(\partial_x)\partial_x + D'(x)\boldsymbol{\mathcal{L}}_-(\partial_x).$$

These conditions effectively imply that energy is conserved at each edge (i.e. no translational or rotational work is done by any of the edges).

3 Integral Equations

The first step in our Wiener-Hopf solution is to use Green's theorem to derive an expression for $\phi(x,z)$. This will lead us to a pair of coupled integral equations—one over the interval $(0,l)$ which generally needs to be solved numerically, and another over (l,∞) which is able to be solved analytically with the Wiener-Hopf technique.

We use a Green's function, G, that satisfies the following set of equations:

$$(\partial_\xi^2 + \partial_\zeta^2 - \alpha_y^2)G(x-\xi,z,\zeta) = \delta(x-\xi,z-\zeta), \tag{4a}$$
$$\boldsymbol{\mathcal{L}}_0(\partial_\xi)G_\zeta(x-\xi,z,0) + G(x-\xi,z,0) = 0, \tag{4b}$$
$$G_\zeta(x-\xi,z,H) = 0, \tag{4c}$$

where $\boldsymbol{\mathcal{L}}_0(\partial_x) = D_0(\partial_x^2 - \alpha_y^2)^2 + \lambda - m_0\mu$.

This Green function depends on the dispersion function for the left-hand region, $f_0(\alpha)$, where $f_j(\alpha) = \coth(\gamma H)/\gamma - \Lambda_j(\gamma)$, $\Lambda_j(\gamma) = \boldsymbol{\mathcal{L}}_j(\mathrm{i}\alpha) = D_j\gamma^4 + \lambda - m_j\mu$ $(j = 0,2)$, and $\gamma(\alpha) = (\alpha^2 + \alpha_y^2)^{1/2}$. In particular, the integral equations depend only on x or ξ derivatives of $g(x-\xi) = G_{z\zeta}(x-\xi,0,0)$, which is given by

$$g(x-\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-\mathrm{i}\alpha(x-\xi)}}{f_0(\alpha)}\mathrm{d}\alpha = \mathrm{i}\sum_{\alpha\in S_1} A_1(\alpha)\mathrm{e}^{\mathrm{i}\alpha|x-\xi|}, \tag{5}$$

where, for $j = 0,2$, $S_j = \{\alpha \mid f_j(\gamma) = 0 \quad \& \quad \mathrm{Im}(\alpha) > 0\}$ and

$$A_j(\alpha) = -(\gamma^2/\alpha)/\big(H(\Lambda_j^2(\gamma)\gamma^2 - 1) + 5D_j\gamma^4 + \lambda - m_j\mu\big).$$

In the limit as ε becomes zero (in the definition of λ), S_j generally contains a positive real root α_j, two complex roots and an infinity of positive imaginary roots. For $\theta < 90°$, $\alpha_y = \alpha_0 \tan\theta$. The effect of ε is to produce a small anti-clockwise rotation of the roots—in particular, this moves the α_j into the upper half-plane.

Applying Green's theorem, coupled with equations (2), (3) and (4), gives us the following integral equation in $\phi_z(x,0)$:

$$\phi_z(x,0) = I\mathrm{e}^{i\alpha_0 x} + \boldsymbol{\psi}^T(x)\,\boldsymbol{P}_0 + \boldsymbol{\psi}^T(x-l)\,\boldsymbol{P}_l + \int_0^\infty \big(\boldsymbol{\mathcal{L}} - \boldsymbol{\mathcal{L}}_0\big)g(x-\xi)\phi_z(\xi,0)\mathrm{d}\xi, \tag{6}$$

where $I = -\gamma_0\tanh\gamma_0 H$, $\boldsymbol{P}_{x_e} = \boldsymbol{\mathcal{P}}(x_e,\partial_x)\phi_z(x_e,0)$, and

$$\boldsymbol{\psi}(x) = -\begin{pmatrix}\boldsymbol{\mathcal{L}}_+(\partial_x)\,\partial_x \\ \boldsymbol{\mathcal{L}}_-(\partial_x)\end{pmatrix} g(x), \quad \boldsymbol{\mathcal{P}}(x,\partial_x) = \big(D(x^+) - D(x^-)\big)\begin{pmatrix}1 \\ \partial_x\end{pmatrix} - \big(D'(x^+) - D'(x^-)\big)\begin{pmatrix}1 \\ 0\end{pmatrix}.$$

Equation (6) can now be written as a pair of coupled integral equations as follows:

$$D_1(x)\phi_1(x) = \sum_{\alpha\in S_0} \left(b(\alpha)\mathrm{e}^{\mathrm{i}\alpha x} + \beta_+(\alpha)\mathrm{e}^{\mathrm{i}\alpha(l-x)}\right) + \fint_0^l (\boldsymbol{\mathcal{L}} - \boldsymbol{\mathcal{L}}_0)g(x-\xi)\phi_1(\xi)\mathrm{d}\xi, \tag{7a}$$

$$\phi_2(x) = \sum_{\alpha\in S_0} \beta_-(\alpha)\mathrm{e}^{\mathrm{i}\alpha(x-l)} + \int_l^\infty (\boldsymbol{\mathcal{L}} - \boldsymbol{\mathcal{L}}_0)g(x-\xi)\phi_2(\xi)\mathrm{d}\xi, \tag{7b}$$

where $\phi_j(x)$ is $\phi_z(x,0)$ restricted to the j^{th} region. The $D_1(x)$ term on the left-hand side of (7a) results from a delta-function type singularity in the kernel of the integral equation; the Cauchy principal value symbol $\fint$ has been used to avoid having to integrate over this singularity at $\xi = x$ numerically. Equation (7a) may be solved with numerical quadrature using the method of Williams & Squire (2004). Since $\boldsymbol{\mathcal{L}}$ is independent of x in the $j = 2$ region, (7b) may be solved by taking a Fourier transform using the Wiener-Hopf technique.

The $\beta^\pm$ coefficients are unknowns that provide the coupling between the two equations. If $b(\alpha) = I\delta_{\alpha,\alpha_0} + iA_0(\alpha)\boldsymbol{p}^T(-\alpha)\boldsymbol{P}_0$, $f_\pm(\alpha) = \gamma^2(\alpha) \pm (1-\nu)\alpha_y^2$, and $\boldsymbol{p}^T(\alpha) = \big(-\mathrm{i}\alpha f_+(\alpha), f_-(\alpha)\big)$, they are given by

$$\beta^+(\alpha) = \mathrm{i}A_0(\alpha)\left(\boldsymbol{p}^T(\alpha)\boldsymbol{P}_l - f_2(\alpha)\int_0^\infty \phi_2(\xi+l)\mathrm{e}^{\mathrm{i}\alpha\xi}\mathrm{d}\xi\right),$$

$$\beta^-(\alpha) = b(\alpha)\mathrm{e}^{\mathrm{i}\alpha l} + \mathrm{i}A_0(\alpha)\left(\boldsymbol{p}^T(-\alpha)\boldsymbol{P}_l + \int_0^l \phi_1(\xi)\left(\boldsymbol{\mathcal{L}} - \boldsymbol{\mathcal{L}}_0\right)\mathrm{e}^{\mathrm{i}\alpha(l-\xi)}\mathrm{d}\xi\right).$$

Now, equation (7b) can be solved analytically, allowing the β_+ coefficients to be written explicitly in terms of the β_-. Hence, the β_+ may be eliminated from (7a), making the latter an integral equation that depends on $\phi_1(x)$ alone, and that can be solved straightforwardly using numerical quadrature.

The final step in the solution is the application of the edge conditions. From (6) and the solution for $\phi_2(x)$, $\phi_z(x,0)$ may be written in the form

$$\phi_z(x,0) = \begin{cases} I\mathrm{e}^{i\alpha_0 x} + \sum_{\alpha\in S_0} a(\alpha)\mathrm{e}^{-i\alpha x} & \text{for } x < 0, \\ \sum_{\alpha\in S_2} d(\alpha)\mathrm{e}^{i\alpha(x-l)} & \text{for } x > l. \end{cases} \tag{8}$$

These expressions may easily be substituted into the definitions of the $\boldsymbol{P}_{x_e}$, giving a system of four linear equations in four variables that must be solved to complete the solution.

4 Conclusions

This method represents a step forward in the treatment of ice scattering problems in that a numerical solution method has been combined with the analytical Wiener-Hopf technique effectively. However, the method does neglect submergence, which could make a difference when very thick ice shelves are modelled and for smaller water depths especially. However, this approach should be able to be generalised in the future to address this deficiency.

References

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'The scattering by a sea ice/ice shelf transition'

Discusser - M.H. Meylan:

Do you believe that the Wiener-Hopf is better that mode-matching at high frequencies?

Reply:

Much the same in terms of number of modes use. Would be faster though, as no matrix to invert in Wiener-Hopf (only a 2×2 matrix for edge conditions for a single edge).

Discusser - R. Porter:

Are there any limits on the thickness of the ice sheet given that you are using thin-plate theory?

Reply:

Should be ok, wavelength is extremely large in relation to wave amplitude ($\lambda > 100m$ for $2s$ period in $20m$ thick ice), and wave amplitude is small in relation to thickness. Also, Balmforth & Craster (2000) and Fox & Squire (?) have more comparisons of thick plate equation versus thin plate equation and there is little difference.