The Amplification and Reflection of Long Gravity Waves (Tsunamis) on the Coastal Rise, and the Flux at the Shoreline

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Introduction

The characteristics of those separate types of energetic waves which appear on the ocean {Tidal, Tsunami; Long Wind Generated} are as distinct as their characteristic periods {12 hrs, 30 minutes, 10 seconds}. So dispersive effects for them are: {absent, very weak, crucial}.

During the trans-ocean propagation of Tsunami waves, weak dispersion results in the appearance of a train of very long waves, as in the decay of a long pulse in very shallow water, Kajiura (1963), see Mei (1983). These travel close to the shallow water limiting speed

$$c = \sqrt{g(h+\eta)} = \sqrt{g\eta^*} \tag{1}$$

or 224 m/s for a water depth, $h = 5$ km, with a wave length, $L_0 = 540$ km, corresponding to a wave period of 40 minutes, as observed in Sri Lanka after the Aceh earthquake, whose maximum trans-ocean wave amplitude, $\hat{\eta}_o$, is estimated at O(1m).

Here we imagine that such a train, $\eta_o(\vec{x}, t)$, approaches the foot of the coastal rise, 100 km from the shoreline in Sri Lanka, while propagating co-directional with the bottom gradient, $\nabla h(\vec{x})$. More than that, we imagine that variations of both the wave and the coastal shape are primarily in the onshore direction of wave propagation, more rapidly at first on the coastal slope and more slowly on the coastal shelf leading to the shoreline.

This very long one-dimensional Tsunami wave becomes transformed on the rise, and amplifies as it progresses shoreward and, as we know, can result in a flood of high water at the shoreline and in a large flux of water across the beach. We would like to understand how this comes about and be able to quantify through suitable formulae the flow speed, u_s, and wave amplitude, $\hat{\eta}_s$, at the shoreline. Since the Tsunami wavelength, L, may be much larger than the length of the rise, ℓ, it is at the same time necessary to understand to what extent the incoming wave is reflected from the rise.

Governing Equations

The governing equations (given here in their full 3-dimensional version for completeness) are based on the Airy (1845) long wave, non-linear approximations:

$$\text{(momentum)} \qquad \vec{u}_t + \underline{\vec{u} \cdot \nabla \vec{u}} + g\nabla\eta = 0 \tag{2}$$

$$\text{(continuity)} \qquad \eta_t + \nabla \cdot [(\underline{\eta} + h)\vec{u}] = 0 \tag{3}$$

where h is the undisturbed water depth, η the elevation of the local water surface above sea level, $\vec{u}$ is the horizontal velocity, assumed uniform in depth, while the vertical velocity v is neglected; the variation of pressure in depth is hydrostatic.

In the case of small wave amplitude, $(\eta/h)^2 << 1$, then the underlined terms in (2) and (3) may be neglected. The resulting linearized theory has been much studied, see Mei (1983). Here, in our Sri Lanka example, we estimate that the linearized theory is applicable for $h > h'$, where:

$$h' = 35\hat{\eta}_T \tag{4}$$

where $\hat{\eta}_T$ is the Tsunami wave amplitude entering the coastal rise, O(1m). In this case, (2) and (3) may be combined into the well known wave equation due to Green (1837), see Lamb (1932):

$$\eta_{tt} - \nabla \cdot (c_\ell^2 \nabla\eta) = 0 \tag{5}$$

$$c_\ell = \sqrt{gh} \tag{6}$$

Green's Theory and Geometrical Optics

Following the assumption that the variation of c over one wavelength, L, of the propagating wave is small

$$\frac{L}{c}\nabla c << 1 \tag{7}$$

Green (1837) has shown that,

$$\hat{\eta}_1/\hat{\eta}_2 = (h_2/h_1)^{1/4} \tag{8}$$

which was later shown, Lamb (1932), to correspond to conservation of the Flux of wave energy, F_E, in shallow water,

$$F_E = (\eta^2/2) \cdot c_\ell = \text{ constant} \tag{9}$$

with the implication that the reflection of wave energy from the bottom may be neglected when (7) applies.

Wave propagation in media with variation of wave speed occurs elsewhere in physics,first of all in Optics, and led to the Eikonal equation and the general theory of Geometric Optics, beginning at the end of the 19th century, see Born and Wolf, (1975). That theory is also based on the approximation (7). Application to the present problem is described by Mei (1983), with results corresponding to (8) and (9). In particular that theory is unable to provide realistic estimates of reflection coefficients, the reflection always being small. Furthermore, it results in unrealistic estimates of the wave height as the shoreline is approached ($h \to 0$). Finally, these theories are **on their face** unsuitable for our problem, since we have in opposition to (7) the condition

$$\frac{L}{c}\nabla c \approx \frac{L}{\ell} > 1 \qquad (10)$$

where at Sri Lanka the Tsunami wave length is 540 km, and the length of the coastal rise is 100 km. Evidently we need a theory capable of dealing with (10) while providing appropriate estimates of wave transformation in the face of reflection.

Method of Characteristics

The exact Airy equations, (2)-(3), form a hyperbolic system and can be solved utilizing the method of characteristics, see Yih (1983). The appropriate non-linear equations corresponding to (2) and (3) are:

$$[\partial/\partial t + (u+c)\partial/\partial x](u+2c) = gh_x \qquad (11)$$

$$[\partial/\partial t + (u-c)\partial/\partial x](u-2c) = gh_x \qquad (12)$$

The two quantities $[(u+2c);(u-2c)]$ can be integrated along the characteristic curves in (x,t) space, defined by ($x_t = u \pm c$). When u and c are known at $x = 0$ for all t, then the solution for all (x,t) behind the leading characteristic, $x_t = c+u$ may be calculated, at least up to the point where separate characteristic lines may merge. These correspond physically to the appearance of wave breaking leading to bore formation, Stoker (1957).

Analyses based on the above method have been carried out by Kishi (1963), see Horikawa (1978). The latter quotes the following result of Kishi's analysis

$$h/h_o = \left\{\frac{\sqrt{1+\hat{\eta}_T/h_o}-1}{\sqrt{1+\hat{\eta}/h}-1}\right\}^{4/5}\left\{\frac{6\sqrt{1+\hat{\eta}_T/h_o}-1}{6\sqrt{1+\hat{\eta}/h_o}-1}\right\}^{6/5} \qquad (13)$$

Taking the limit when $h << \hat{\eta}$, approaching the shoreline, we find, remarkably, that (13) takes the asymptotic form:

$$(h/\hat{\eta} \to 0) \qquad \hat{\eta}_s/\hat{\eta}_T = (.40)(h_o/\hat{\eta}_T)^{1/5} \qquad (14)$$

a simple formula for the wave elevation, $\hat{\eta}_s$, at the shoreline, and in form, in agreement with the results derived below through an alternative analysis.

Alternative Analysis Based on Co-ordinate Straining

We take advantage of the linearity of the system over most of the rise, and then treat the peak of the wave motion in the non-linear region separately through a simple extension of the linear results into the weakly non-linear regime.

We begin with the following non-linear equations which are fully equivalent to the Airy equations, (1)-(3), and are obtained from the latter by simple manipulations,

$$\vec{q}_{tt} - (g\eta^*)\nabla^2\vec{q} = -R \qquad (15);$$

$$R = \{\vec{u}\nabla\cdot\vec{q}_t + (2\vec{u}_t + \vec{u}\nabla\vec{u})\nabla\cdot\vec{q} + (\nabla\vec{u})\cdot\vec{q}_t + \vec{q}\cdot\nabla\vec{u}_t\} \qquad (16);$$

$$\eta^*_t + \nabla\cdot\vec{q} = 0 \qquad (17);$$

$$\vec{q} = \vec{u}\eta^* \qquad (18);$$

$$\eta^* = h + \eta \qquad (19)$$

If the LHS of (15) is taken as 0(1), then

$$R = 0(u/c) = 0(\eta/h) \qquad (20)$$

and therefore on the rise, the following one-dimensional wave equation pertains with the variable wave speed, $\sqrt{g\eta^*} \approx \sqrt{gh} = c_\ell$:

$$q_{tt} - c_\ell^2 q_{xx} = 0 \qquad (21)$$

for all depths less than h', see (4). For the solution of (21) appropriate to our situation here we adopt a new co-ordinate ξ, defined by the straining relation

$$c_\ell \xi_x = c_o \qquad \text{(a constant, } \sqrt{gh_o}) \qquad (22)$$

so that (21) is transformed into a damped wave equation with constant wave speed, c_o, in the strained co-ordinate,

$$q_{tt} - c_o^2 q_{\xi\xi} + \underline{c_o(c_\ell)_x \cdot q_\xi} = 0 \qquad (23)$$

and,

$$\eta_t = -(c_o/c_\ell)q_\xi \qquad (24)$$

This formulation offers two obvious advantages over its parent, (21). First, the variation in wave speed in physical coordinates has been absorbed in the change of variable, and provides the wave part of the solution without ambiguity — although the effect of the damping term, underlined in (23), remains to be determined; furthermore, the constancy of the wave speed informs us that no shocks or bores may appear. Second, the very appearance of the damping term informs us immediately that propagating waves are possible only for sufficiently small gradients in c_ℓ, while for larger gradients complete reflection of the incoming wave will occur. As we shall see, this allows us to deal realistically with (7).

If we replace $c_o(c_\ell)_x$ in (23) by its spatial average on the rise, then

$$q_{tt} - c_o^2 q_{\xi\xi} - c_o^2/\ell \cdot q_\xi = 0 \tag{25}$$

which yields a useful approximation provided that $c_o(c_\ell)_x$ does not vary too much. Fortuitously this damping factor is **constant** for a particular bottom slope which is at the same time an excellent model of a generic rise! It is,

$$c_\ell^2/g = h = h_o[1 - x/\ell]^2 \tag{26}$$

which has a slope at the beginning of the rise ($x = 0$), $h_x = 2h_o$, which declines by half at $x/\ell = 1/2$ and vanishes slowly on the shelf leading to the surf line at $x = 0$.

For this model, (26), then in view of (22), the damping factor becomes,

$$c_o^2(c_\ell)_\xi/c_\ell = -c_o^2/\ell \tag{27}$$

so that, corresponding to (26),

$$c_\ell = c_o e^{-\xi/\ell} \tag{28}$$

The General Solution and Amplification on the Generic Rise

The general solution of (25) in Fourier form consists of the sum of two independent waves on the rise, one traveling toward the shore (−), and the other seaward (+):

$$q_r(\xi, t) = \Sigma_j \int_{-\infty}^{+\infty} A_j(\sigma) e^{-\xi/2\ell} e^{i[\sigma t \pm \kappa \xi]} d\sigma \tag{29}$$

to be summed over j, where $j = -$, corresponds to the upper signs,and $j = +$, to the lower, and where $\kappa = \sqrt{\beta^2 - 1}/2\ell$, and

$$\beta = 2\sigma\ell/c_o = 4\pi(\ell/L) \tag{30}$$

Correspondingly, making use of (24), the wave elevation, η_r, may be calculated from q_r:

$$\eta_r(\xi, t) = \Sigma_j \int_{-\infty}^{+\infty} (A_j/c_o)\gamma_j(\beta) e^{\xi/2\ell} e^{i[\sigma t \mp \sqrt{\beta^2-1}(\xi/2\ell)]} d\sigma \tag{31}$$

where

$$\gamma_j(\beta) = \left[\frac{\pm\sqrt{\beta^2 - 1} - i}{\beta}\right] \tag{32}$$

The two amplitude spectra (A_-, A_+) are found by matching η_r and q_r with the incoming Tsunami wave at $(x, \xi = 0)$. where,

$$\eta_T(x, t) = \int_{-\infty}^{+\infty} A_T(\sigma) e^{i[\sigma t - \kappa_T x]} d\sigma \tag{33}$$

and

$$q_T = h_o u_T = c_o \eta_T \tag{34}$$

with the results,

$$c_o A_T = \Sigma_j A_j = A_- + A_+ \tag{35}$$

$$c_o A_T = \Sigma_j A_j \left[\frac{\pm\sqrt{\beta^2 - 1} - i}{\beta}\right] \tag{36}$$

and finally,

$$A_j = \pm c_0 A_T \left[\frac{\beta \pm \sqrt{\beta^2 - 1} + i}{2\sqrt{\beta^2 - 1}}\right] \tag{37}$$

For a suitably long coastal rise ($\underline{\beta >> 1}$), $A_+ \to 0$, so,

$$\eta_r = \eta_-(\xi, t) = \int_{-\infty}^{+\infty} A_T(\sigma) e^{\xi/2\ell} e^{i[\sigma t - \kappa_T \xi]} d\sigma \tag{38}$$

showing that the Tsunami shape is preserved on the rise in strained coordinates (ξ), while amplifying like $e^{\xi/2\ell}$.

Since $e^{\xi/2\ell} = \sqrt{c_o/c_\ell}$, it follows from (38) that,

$$(\hat{\eta}_r)^2 \cdot c_\ell = A_T^2 \cdot c_o \qquad \text{constant} \tag{39}$$

so that the **flux of wave energy**, see (9), is again **conserved**. Therefore, the latter conservation law is a much more general result than the limitations of geometrical optics suggests.

Wave Reflection on the Generic Rise

As the coastal rise shortens it eventually becomes effective in reflecting the incoming long wave, and finally for $\beta \leq 1$ or $\ell \leq L/4\pi$ only standing waves exist:

$$c_o \eta_r(\xi, t) = \Sigma_j \int_{-\infty}^{+\infty} A_j \gamma_j(\beta) e^{\xi/2\ell(1 \pm \sqrt{1-\beta^2})} e^{\sigma t} d\sigma \tag{40}$$

At $\underline{\beta = 1}$, the two waves are large but almost cancel each other with the result,

$$\eta_r(\xi, t) = \int_{-\infty}^{+\infty} A_T e^{\xi/2\ell} e^{i\sigma t} d\sigma \tag{41}$$

so that $\hat{\eta}_r$ in this case is identical to the case of the long rise, (38).

For shorter rises, A_+ exceeds A_-, and as $\beta \to 0$, A_- can be neglected and,

$$\eta_r \to \int_{-\infty}^{+\infty} A_T e^{i\sigma t} d\sigma \to \eta_T(0, t) \tag{42}$$

constant on the rise, without amplification! Also $q_r \to q_T(0, t)$. So the amplification of the wave due to the rise decreases as β decreases from the value unity, and disappears entirely for very short rises.

We note that this theory does not take into account any reflection which might take place at the shoreline itself, from the presence there of a seawall, for instance. In that case we might expect further effects which travel seaward and modify the incoming Tsunami, as in the normal reflection of an acoustic wave from a wall.

The Onshore Amplification Law

Closest to the shore, strong non-linear effects and bottom irregularities can occur, resulting in the creation of bores. In this regime, say for $h < \eta$,the oncoming flow, characterized by high velocities, more resembles an open channel flow than an ocean wave. Seaward of the bore which may form, lies the amplified wave, pushing water ahead of it into the open-channel regime approaching the shore. It is the height of this amplified wave which we seek to estimate, as it eventually appears at the shoreline behind the bores and pours onto the land beyond the shore.

In the likely absence of any further amplification of this wave in the region closest to shore, $h < \eta$, we seek predictions based upon an extension of the linear regime into the weakly non-linear regime, i.e. the region where,

$$\hat{\eta} < h < 35\hat{\eta}_T \tag{43}$$

In this regime we apply the wave equation, (21), except that we replace the linear wave speed, $c_\ell = \sqrt{gh}$, by its non-linear counterpart, $c = \sqrt{g\eta^*}$, as in the exact parent equation (15). We neglect, leaving justification for future analysis, the non-linear residual, R.

After suitably modifying (28) we find,

$$c_s = \sqrt{g\hat{\eta}_s} = c_o e^{-\xi_s/\ell} \qquad 44$$

so that the effective shoreline is found at a finite value of ξ, ξ_s, rather than at infinity, as in the linearized treatment. For the amplitude at the shore, we utilize (31)-(37) as before. Then, for suitably long rises, $\beta >> 1$, as in Sri Lanka, we utilize (38) so that,

$$\hat{\eta}_s = \hat{\eta}_r e^{\xi_s/2\ell} \tag{45}$$

Combining (44) and (45), eliminating ξ_s, a new Amplification Law can be found,

$$\hat{\eta}_s/\hat{\eta}_T = (h_o/\hat{\eta}_T)^{1/5} \tag{46}$$

For example: $h_o = 5$ km; $\hat{\eta}_T = 1$m; $\hat{\eta}_s/\hat{\eta}_T = 5.6$; $\hat{\eta}_s = 5.6$ m. This height of the onshore wave is consistent with anecdotal observations in Sri Lanka. This result, (46), corresponds to (9), conservation of wave energy flux, where c_ℓ is replaced by c.

For the speed onshore, using previous results,

$$u_s = q_s/\eta_s = \left(ce^{-\xi_s/2\ell}\hat{\eta}_T\right)/e^{\xi_s/2\ell}\hat{\eta}_T \tag{47}$$

or,

$$u_s = c_s \tag{48}$$

so that Froude number on shore, F_s, is unity:

$$F_s = u_s/c_s = 1 \tag{49}$$

This result happens to correspond, perhaps accidentally, to the local hydraulic law which pertains for open channel flow spilling over a barrier, as at the top of a dam spillway, see Chow (1961). Recall that all these results and conclusions apply for sufficiently long rises, $\beta << 1$. Other results and conclusions may be found from the preceding analysis where strong reflections from the rise dominate.

Finally, we note that the Amplification Law (46) corresponds in form to (14), the result we have found from Kishi's formula (18) in the limit, $h << \hat{\eta}$, i.e. approaching the shoreline. This correspondence, as it was obtained by a completely different approach, lends support to the present analysis. The appearance of the factor (.40) in (14), missing in (46), may result from reflections from the rise in the case considered by Kishi.

We have largely fulfilled our original purpose, to understand and predict the amplification of Tsunamis on the coastal rise. Strongly non-linear effects close to the shoreline, including the appearance of bores, remain to be elucidated.

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