Resonant effects in scattering by periodic arrays

I. Thompson[†*] **and C. M. Linton**[†]
[†]Department of Mathematical Sciences, Loughborough University, Loughborough, Leics. UK
*Email: i.thompson@lboro.ac.uk

Introduction

This paper is concerned with resonant effects caused by the scattering of a plane wave by a periodic array of vertical circular cylinders. The scattered field is known to consist of a finite set of plane waves, and an infinite number of evanescent modes which decay exponentially as the perpendicular distance between array and observer is increased. The problem is said to be resonant when the direction of propagation of a scattered plane wave is exactly parallel to the array. An efficient method for computing the solution in this special case is presented. No such method appears to exist in the literature; other authors have derived only asymptotic results as the resonant state is approached (see Twersky [1], for example).

Analysis

Consider a periodic array of vertical circular cylinders of radius a standing in a fluid of constant depth h. The fluid extends to infinity in the x and y directions, and the cylinders extend throughout its depth. The axis of cylinder p is located at $(ps, 0)$, $p \in \mathbb{Z}$ in the (x, y) plane. Let the plane wave

$$\phi^{\mathrm{i}} = \Re[\mathrm{e}^{\mathrm{i}k(x\cos\psi_0 + y\sin\psi_0)}\mathrm{e}^{-\mathrm{i}\omega t}]\cosh[k(z+h)] \tag{1}$$

be incident at angle ψ_0 upon the array. The depth and time dependence of all fields is identical to (1); henceforth we omit the factors $\mathrm{e}^{-\mathrm{i}\omega t}$ and $\cosh[k(z+h)]$, and also the symbol $\Re$. We are left with the two dimensional scattering problem shown in figure 1, in which all fields $\phi(x, y)$ satisfy the Helmholtz equation

$$(\nabla^2 + k^2)\phi = 0.$$

The total field is expressed in the form

$$\phi^{\mathrm{t}} = \phi^{\mathrm{i}} + \phi^{\mathrm{s}},$$

where ϕ^{s} is the scattered response from the array. Introducing shifted sets of polar co-ordinates (r_p, θ_p) with the origin positioned at the centre of scatterer p, the boundary condition on the surface of the scatterers requires that

$$\mathrm{d}\phi^t/\mathrm{d}r_p = 0 \tag{2}$$

on $r_p = a$. Exploiting the fact that the only difference between the field at $x = x_0$ and that at $x = x_0 + ps$ is the phase factor $\mathrm{e}^{\mathrm{i}kps\cos\psi_0}$, we represent the scattered field in the form

$$\phi^{\mathrm{s}} = \sum_{p=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kps\cos\psi_0} B_n \,\mathrm{H}_n(kr_p)\mathrm{e}^{\mathrm{i}n\theta_p}, \tag{3}$$

where H_n is the nth order Hankel function of the first kind. An infinite system of equations for the unknown constants B_n can be obtained by applying boundary conditions on the surface of the scatterers using Graf's addition theorem [2] as in Linton & McIver [3]. Thus, we have

$$B_m + Z_m \sum_{n=-\infty}^{\infty} B_n \sigma_{n-m} = -Z_m \mathrm{i}^m \mathrm{e}^{-\mathrm{i}m\psi_0}, \quad m \in \mathbb{Z}, \tag{4}$$

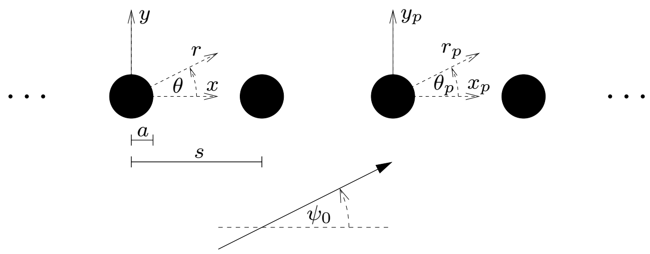

Figure 1: The infinite array, with scatterers centred at $(ps, 0)$ for integer p, and with a plane wave incident at angle ψ_0.

wherein

$$\sigma_n = \sum_{j=1}^{\infty} [\mathrm{e}^{-\mathrm{i}jks\cos\psi_0} + (-1)^n \mathrm{e}^{\mathrm{i}jks\cos\psi_0}]\, \mathrm{H}_n(jks). \tag{5}$$

For the Neumann condition (2), the coefficients Z_n take the form

$$Z_n = \mathrm{J}_n'(ka)/\,\mathrm{H}_n'(ka). \tag{6}$$

As $|n| \to \infty$, the sequence $\{Z_n\}$, and therefore $\{B_n\}$, converges to zero exponentially [2]. The quantity σ_n is known as a Schlömilch series, its value can be computed efficiently using expressions given in [4].

Now, introduce a small damping factor by writing $k = k_r + \mathrm{i}\epsilon$, where $\epsilon > 0$, $k_r \in \mathbb{R}$, and insert the integral representation [5]

$$\mathrm{H}_n(kr)\mathrm{e}^{\mathrm{i}n\theta} = \frac{(-\mathrm{i})^{n+1}}{\pi} \int_{-\infty}^{\infty} \left[\frac{\alpha - \gamma(\alpha)}{k}\right]^{n\,\mathrm{sgn}(y)} \mathrm{e}^{-\gamma(\alpha)|y| + \mathrm{i}\alpha x}\, \frac{\mathrm{d}\alpha}{\gamma(\alpha)}$$

into (3). Here $\gamma(\alpha) = (\alpha^2 - k^2)^{1/2}$, with $\gamma(0) = -\mathrm{i}k$, and the damping factor moves the branch points off the real line. Taking the limit $\epsilon \to 0$ determines the direction in which the integration contour should be indented in order to obtain the time-harmonic solution. The Poisson summation formula then yields

$$\phi^{\mathrm{s}} = \sum_{j=-\infty}^{\infty} A_j^{\pm}\, \mathrm{e}^{\mathrm{i}k(x\cos\psi_j + |y|\sin\psi_j)}, \tag{7}$$

where the scattering angles ψ_j are given by

$$k\cos\psi_j = k\cos\psi_0 + 2j\pi/s, \tag{8}$$

and the amplitude coefficients by

$$A_j^{\pm} = \frac{2}{ks} \sum_{n=-\infty}^{\infty} (-\mathrm{i})^n B_n \mathrm{e}^{\pm\mathrm{i}n\psi_j} \csc\psi_j. \tag{9}$$

Here, the upper and lower signs refer to $y > 0$ and $y < 0$, respectively, and the expression (7) is valid everywhere except on $y = 0$. A finite number of plane waves are included; note that $\gamma(k\cos\psi_j) = -\mathrm{i}k\sin\psi_j$, therefore terms for which $|\cos\psi_j| > 1$ represent evanescent modes, which can be neglected for large $|y|$. Resonance occurs if $\sin\psi_p = 0$ for some $p \in \mathbb{Z}$ since then there is a plane wave propagating parallel to the array. In this case, $A_p^{\pm}$ cannot be obtained directly from (9), nor, in fact can B_n be computed from (4) since the Schlömilch series (5) is now divergent.

Nevertheless, all physical quantities must remain finite, and can be determined as follows. We restrict our attention to the case in which $\psi_p = 0$; $\psi_p = \pi$ is equivalent to $\psi_{-p} = 0$ with the incidence angle ψ_0 replaced by $\pi - \psi_0$. The case of double resonance, in which $\psi_p = 0$ and $\psi_q = \pi$ with $p, q \in \mathbb{Z}$, can occur when $ks = (p - q)\pi$, however this is not considered here. Results in [4] show that

$$\sigma_n = \hat{\sigma}_n + 2(-\mathrm{i})^n/(ks\psi_p), \tag{10}$$

where $\hat{\sigma}_n$ remains bounded as $\psi_p \to 0$. Thus, the crucial step is to consider the Schlömilch series σ_n and the coefficient B_n *as functions of the scattering angle* ψ_p. The situation can then be explained in the following manner. The Schlömilch series $\sigma_n(\psi_p)$ has a simple pole at the point $\psi_p = 0$, however the total residue obtained from the summation in (4) must be zero in order to yield a finite right hand side. Hence we can introduce the Taylor expansion

$$\frac{2}{ks} \sum_{n=-\infty}^{\infty} (-\mathrm{i})^n B_n(\psi_p) = a_1 \psi_p + O(\psi_p^2), \tag{11}$$

then on using (10) in (4) and evaluating at $\psi_p = 0$, we obtain

$$B_m(0) + Z_m \sum_{n=-\infty}^{\infty} B_n(0)\hat{\sigma}_{n-m}(0) = -Z_m \mathrm{i}^m [\mathrm{e}^{-\mathrm{i}m\psi_0} + a_1]. \tag{12}$$

The presence of the extra unknown a_1 necessitates the use of an additional equation to close the system; this is obtained by evaluating (11) at $\psi_p = 0$. Expanding $\mathrm{e}^{\mathrm{i}n\,\mathrm{sgn}(y)\psi_p}$ in powers of ψ_p, and using (11) in (9), we can now take the limit $\psi_p \to 0$ to obtain

$$A_p^{\pm} = a_1 \pm \frac{2\mathrm{i}}{ks} \sum_{n=-\infty}^{\infty} n(-\mathrm{i})^n B_n(0). \tag{13}$$

In the case $p = 0$ (grazing incidence), the entire system can be solved by inspection; we find that $B_n(0) = 0$ and $a_1 = -1$. Thus in the limit $\psi_0 \to 0$, the scattered response eliminates the incident wave, i.e. the array behaves as a simple Dirichlet boundary located on $y = 0$.

Results & Discussion

The system of equations (4) and its resonant counterpart (12) can be solved numerically by truncation. Figure 2 shows two mode amplitudes for the total field above and below the array for $ks = 2.5$, $ka = 0.5$ with varying ψ_0. The quantities $1 + A_0^+$ and A_0^- are the transmission and reflection coefficients for the array; note the symmetry about $\psi_0 = \frac{\pi}{2}$ for these plots. Equation (4) was used for the continuous curves. Sharp changes in gradient occur close to resonances; there are six in this case, one for each of the modes $j = \pm(1, 2, 3)$. Thus the amplitude of each mode is affected by resonances in each other mode. The locations of the cross symbols $(\times)$ were determined by solving (12), with $p = 1$. These indicate the correct value for $A_1^{\pm}$ when this mode is resonant. Note the rapid decrease in both $|A_1^+|$ and $|A_1^-|$ as mode 1 changes from an evanescent wave (to the left of the $\times$ symbols) to a plane wave (to the right). Also note that, at its resonance, mode 1 dominates above the array, since all higher modes are evanescent. Despite the use of one thousand data points for each plot, the curve for $|A_1^+|$ fails to capture the correct behaviour at resonance. Testing the behaviour of the solution on the surface of the scatterers using the boundary condition indicates that the resonant calculation using (12) is correct; it is difficult to accurately compute the amplitude by setting ψ_0 close to the critical angle.

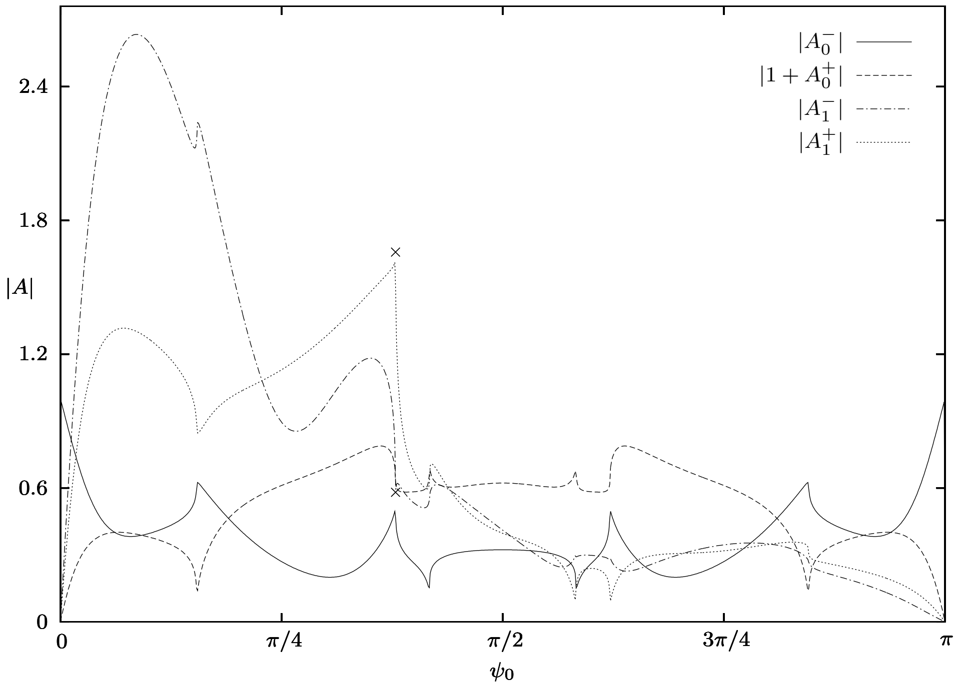

Figure 2: Mode amplitudes for $ks = 2.5$, $ka = 0.5$. The cross symbols $(\times)$ indicate the location of the resonance for mode 1.

References

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