Motion trapping structures in the three-dimensional water-wave problem

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Introduction

Within the linearized inviscid theory of water waves, certain structures when held fixed can support a trapped mode of a particular frequency [1]; this is a free oscillation of an unbounded fluid with a free surface. A trapped mode has finite energy, does not radiate waves to infinity, and will persist for all time. The significance of such a mode is that if, for a specified frequency of oscillation, the structure does *not* support a trapped mode, then the solutions to the radiation and scattering problems at that frequency are unique. Trapped modes supported by a fixed structure cannot be excited when that structure is allowed to float freely and hence respond to the hydrodynamic forces acting upon it [2]. In particular, modes cannot be excited in the time domain by any incident wave or by giving the structure an initial displacement and/or velocity. This observation led to a search for *freely-floating structures* that are able to support trapped modes, and to their discovery in the two-dimensional water-wave problem [3]. To distinguish this new mode, a free oscillation that involves the coupled motion of the fluid and structure will be called a "motion trapped mode" and the corresponding structure a "motion trapping structure". The significance of a motion trapped mode is that if, for a specified frequency of oscillation, the structure does not support such a mode, then there is a unique solution to a problem in which the structure is able to float freely.

The existence of a motion trapped mode with a frequency $\omega = \omega_0$ requires the corresponding hydrodynamic coefficients to satisfy two conditions at this frequency. First of all the damping must be zero, and secondly the added mass value must ensure that the equation of motion for the structure has non-trivial solutions in the absence of any forcing. Two-dimensional structures that meet these requirements have already been constructed [3]. Here the construction is extended to three dimensional heaving structures with a vertical axis of symmetry.

Formulation

Cartesian coordinates x, y, z are chosen with z directed vertically upwards from the mean free surface. Also, R, θ are used to denote spherical coordinates with θ measured from the downward vertical, and $r = R\sin\theta$ to measure horizontal distance from the z axis.

Consider a surface-piercing structure constrained to move freely in the vertical direction with initial displacement $Z(0)$ and initial velocity $\dot{Z}(0)$. In the absence of moorings, the Fourier transform of the time-domain equation of motion yields the frequency-domain equation

$$\left[\rho g W - \omega^2 \left\{M + a(\omega) + \mathrm{i}b(\omega)/\omega\right\}\right] v(\omega) = -\mathrm{i}\omega[X(\omega) + M\,\dot{Z}(0)] - \rho g W\, Z(0). \quad (1)$$

Here ρ is the fluid density, g is the acceleration due to gravity, W is the water-plane area, M is the structure's mass, a is the added mass, b is the damping, v is the (complex) amplitude of the structural velocity, and X is the exciting force. There is a unique solution of (1) for $v(\omega)$ if

$$\left[\rho g W - \omega^2 \left\{M + a(\omega) + \mathrm{i}b(\omega)/\omega\right\}\right] v = 0 \quad (2)$$

has only the trivial solution $v = 0$. Necessary conditions for the existence of a non-zero v, corresponding to a motion trapped mode, for some particular frequency $\omega = \omega_0$ are that

$$\rho g W - \omega_0^2 \left\{M + a(\omega_0)\right\} = 0 \quad \text{and} \quad b(\omega_0) = 0 \quad (3)$$

(the latter implies that the oscillating structure does not radiate waves). Kyozuka & Yoshida [4] constructed wave-free structures with $b(\omega_0) = 0$ but, in general, such a structure does not

possess hydrodynamic characteristics that allow the first of equations (3) to be satisfied at $\omega = \omega_0$. However, as shown below, this can be achieved for particular structures.

Denote by ϕ_0 the velocity potential corresponding to the vertical oscillations of a wave-free structure at frequency $\omega = \omega_0$ so that, in particular,

$$\frac{\partial \phi_0}{\partial n} = n_z \quad \text{on} \quad \Gamma \tag{4}$$

where n is a normal coordinate to the wetted surface of the structure Γ, directed out of the fluid, and n_z is the vertical component of the unit normal to Γ. For vertically axisymmetric three-dimensional motion in fluid of infinite depth any wave-free potential must satisfy

$$\phi_0 = \frac{\mu \cos\theta}{R^2} + o\left(\frac{1}{R^2}\right) \quad \text{as} \quad R \to \infty, \tag{5}$$

where μ is a constant and, in general, to leading order ϕ_0 is dipole-like at infinity. Let S denote the union of Γ with the free surface F and a closing hemisphere S_∞ at infinity in $z < 0$. Green's theorem applied over S to ϕ_0 and $u = z + 1/K$, $K = \omega_0^2/g$, yields after some manipulation

$$\rho g W - \omega_0^2 \{M + a(\omega_0)\} = -2\pi\mu\rho\omega_0^2 \tag{6}$$

so that the first condition in (3) may be satisfied if and only if the dipole coefficient μ is zero.

Construction of motion trapping structures

The method of Kyozuka & Yoshida [4] yields structures with zero damping by seeking suitable streamlines of the flows generated from wave-free potentials. The same method is used here with the additional requirement that the wave-free potential ϕ_0 has a far-field dipole coefficient $\mu = 0$. From equation (4), if a suitable structure could be identified, the modified potential

$$\phi = z - \phi_0 \tag{7}$$

would satisfy

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad \Gamma \tag{8}$$

and Γ is a stream surface of the flow corresponding to ϕ; motion trapping structures are obtained from any stream surface of the flow that isolates the singularities of ϕ from infinity. Here the construction uses solutions that are singular on a circular ring of radius c in the free surface. A wave-free ring source $\phi_{\rm s}$ and a potential $\phi_{\rm d}$ that is dipole-like at infinity are combined as

$$\phi_0 = \delta(\phi_{\rm s} + \sigma\phi_{\rm d}), \tag{9}$$

where δ is a free parameter and σ is chosen to ensure that ϕ_0 has no far-field dipole component. For $Kc = j_{0n}$, where j_{0n} is the nth zero of the Bessel function J_0, a wave-free ring-source potential is

$$\phi_{\rm s} = 8c \int_0^\infty (\mu\cos\mu z + K \sin\mu z) I_0(\mu r_<) K_0(\mu r_>) \frac{\mu \,{\rm d}\mu}{\mu^2 + K^2}, \tag{10}$$

where I_0 and K_0 are modified Bessel functions, $r_< = \min\{r, c\}$ and $r_> = \max\{r, c\}$. C. M. Linton (private communication) has derived a set of wave-free potentials that are singular on a ring in the free surface and dipole-like in the far field. The one used here is

$$\phi_{\rm d} = (\cosh\alpha - \cos\beta)^{1/2} \Big\{ Kc\, P_{\frac{1}{2}}(\cosh\alpha)\sin\beta + P_{\frac{1}{2}}(\cosh\alpha)\cos\beta \\ - \tfrac{1}{4} P_{-\frac{1}{2}}(\cosh\alpha) - \tfrac{3}{4} P_{\frac{3}{2}}(\cosh\alpha)\cos 2\beta \Big\}, \tag{11}$$

where P_ν is an associated Legendre function, and (α, β) are toroidal coordinates defined though

$$r = \frac{c\sinh\alpha}{\cosh\alpha - \cos\beta} \quad \text{and} \quad z = -\frac{c\sin\beta}{\cosh\alpha - \cos\beta}, \quad 0 \le \alpha < \infty, \quad 0 \le \beta \le \pi. \tag{12}$$

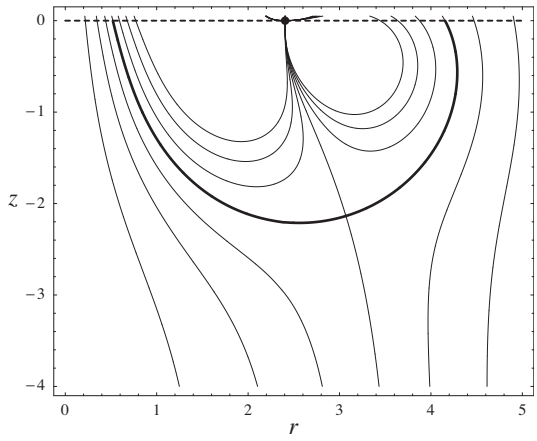

Figure 1: Stream surfaces corresponding to (13); $K = 1$, $c = j_{01}$, $\delta = 1$.

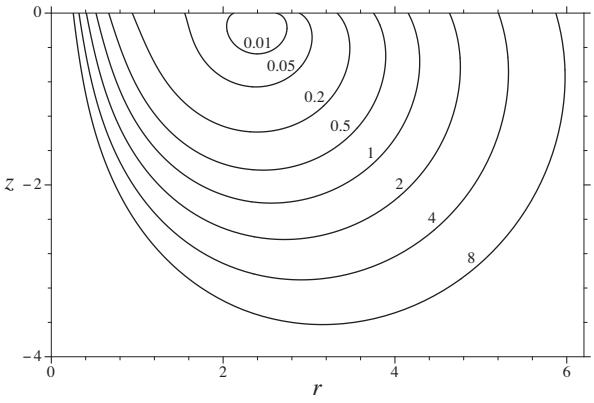

Figure 2: Structural surfaces for $K = 1$, $c = j_{01}$ and various δ.

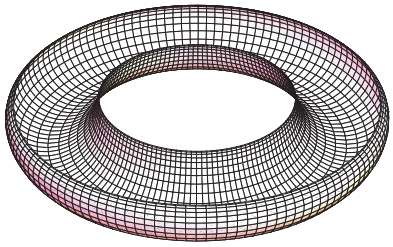

Figure 3: The submerged surface of the trapping structure corresponding to $\delta = 0.05$ in figure 2.

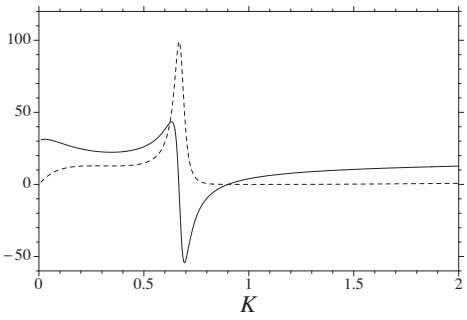

Figure 4: Heave added mass (———) and damping (———) for the structure shown in figure 3.

The required stream surfaces are the level surfaces of

$$\psi = -\tfrac{1}{2}r^2 - \delta(\psi_{\rm s} + \sigma\psi_{\rm d}), \tag{13}$$

where $\psi_{\rm s}$ and $\psi_{\rm d}$ are the Stokes stream functions corresponding to $\phi_{\rm s}$ and $\phi_{\rm d}$. On $r = 0$,

$$\phi_{\rm s} \sim -\frac{4\pi c}{Kz^2} \quad \text{and} \quad \phi_{\rm d} \sim \frac{2^{3/2}Kc^3}{z^2} \quad \text{as} \quad z \to -\infty \tag{14}$$

so that the choice $\sigma = \sqrt{2}\pi/(K^2c^2)$ ensures that ϕ_0 has no far-field dipole component.

Here the choice $Kc = j_{01} \approx 2.40$ is made and a typical stream-surface pattern is shown in figure 1. In $z < 0$ a dividing stream surface separates the flow connected to large depths from that entering the singularity. Part of this stream surface, shown as a thick line in the figure, corresponds to the surface of a motion trapping structure. For a given ϕ_0 there is a single suitable stream surface that isolates the singular ring in a suitable way. However, as the parameter δ is varied a family of motion trapping structures is obtained and some examples are shown in figure 2; a perspective view of one of these structures in figure 3.

Consequences of the existence of motion trapped modes

If one or both of the initial displacement $Z(0)$ and velocity $\dot{Z}(0)$ are non zero then from (1) the velocity v will have a simple pole at $\omega = \omega_0$ and the solution to the boundary-value problem will not exist at $\omega = \omega_0$. However, if $Z(0) = \dot{Z}(0) = 0$ then a zero in X annuls the zero arising from (3) and $v(\omega)$ is non-singular at the resonant frequency so that the solution exists, although it will not be unique because any multiple of the motion trapped mode can be added to the

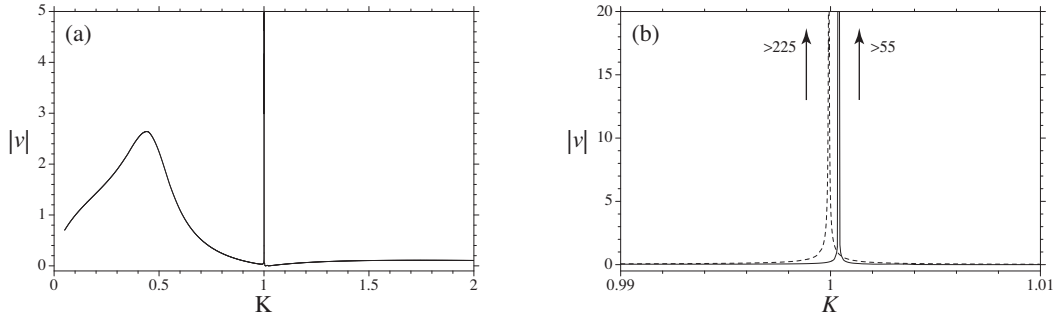

Figure 5: Vertical velocity v excited by an incident wave for the structure shown in figure 3. In (b) comparison is made between WAMIT calculations with 8704 (– – – –) and 2176 (———) panels on a quarter of the wetted surface (additional free-surface panels were used to eliminate irregular frequencies).

solution. In numerical calculations this behaviour will not be captured precisely and there will be a pole of $v(\omega)$ close to the positive real ω axis in the lower half of the complex frequency domain. The velocity of the structure undergoes very rapid changes as a function of frequency, but the hydrodynamic coefficients are well behaved in the vicinity of $\omega = \omega_0$. The added mass and damping coefficients corresponding to the structure in figure 3 are shown as a function of frequency in figure 4. The rapid variations in added mass and damping associated with sloshing in the moonpool occur well away from the motion resonance at $K = \omega^2/g = 1$. It is typical of the motion trapping structures generated by the present method that the damping coefficient is very close to zero over quite a wide range of frequencies around the resonant frequency.

Figure 5 shows the vertical velocity as a function of frequency when the structure is excited by incident waves of unit amplitude. A "spike" arising from the motion trapping is evident near $K = 1$. However figure 5(b) shows that the spike in the velocity is discernible only in a very narrow band which could easily be missed, and that the spike moves closer to $K = 1$ as the number of panels on the wetted surface of the structure is increased. It is difficult to compute reliably near the resonant frequency because, when solving the equation of motion for $v(\omega)$, it is necessary to evaluate the ratio of two quantities that are both very close to zero.

McIver and McIver [3] examined the behaviour of two-dimensional motion trapping structures in the time domain and found that if such a structure is initially at rest in its equilibrium position, then the motion trapped mode cannot be excited by an incident wave packet. This is a direct consequence of the zero in the exciting force at $\omega = \omega_0$. However, if the structure is given an initial non-zero displacement or velocity then the mode will be excited so that after an initial transient has died away the structure and fluid motions settle to oscillations at the trapped-mode frequency. Similar conclusions apply to the three-dimensional structures obtained here.

References

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