Subharmonic resonance of a trapped wave near a vertical cylinder in a channel

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Summary. It is known that perfectly trapped surface waves exist at certain eigen frequencies near a vertical cylinder in a long channel, or an infinite and periodic array of vertical cylinders, and excitation by incident waves of the same frequency is not possible according to the linear theory. We present a nonlinear theory whereby a trapped wave near a cylinder in a channel is excited subharmonically by an incident wave of twice the eigen frequency.

The linearized theory of water-wave trapping either by a stationary body in a channel or by an infinite and periodic array of fixed bodies has been extensively treated by Evans and his associates in the past decade (See Evans & Linton, 1991), Callan et al. (1991), Evans & Porter (1999). Maniar & Newman (1997), Linton & Evans, 1992a), Evans et al (1994), Evans & Porter (1997, 1998) and Utsunomiya & Eatock Taylor (1999)). The occurence of these modes around a multi-legged structure such as an offshore airport may pose a threat to the safety of the installation, hence is of engineering interest. According to the linearized theory, perfectly trapped modes cannot be resonated by incident waves of the same frequency, since no propagation is possible below cut-off. Only in the case of a finite number of periodically spaced cylinders in an infinite sea, is there no cut-off, and trapping is imperfect. Synchronous resonance can be predicted by a linear theory as in Maniar & Newman (1997). However, amplification at resonance is found to increase with the number of cylinders, when real fluid effects are not included.

In coastal oceanography it is known that trapped edge waves can also be present on a sloping beach. In the ideal case of an infinite and uniform beach, trapping is also perfect. Though not synchronously by the incident sea, an edge wave can be resonated subharmonically by incident waves of twice the frequency, as studied by Guza & Davis (1974), Guza & Bowen (1976), Minzoni & Whitham (1977) and Rockliff (1978). Of more recent interest in coastal engineering is the case of mobile barriers for protecting Venice Lagoon from storm tides. Experiments have revealed that normally incident sea waves can force the neighboring gates to oscillate in opposite phases, at half the frequency. The cause for this oscillation was later found to be the existence of trapped modes owing to the periodic and mobile construction Mei et al, 1994). A nonlinear theory for monochromatic incident waves, similar to the subharmonic resonance of edge waves, has been given by Sammarco et al. (1997a), and confirmed by laboratory experiments. Extension to narrow-banded incident waves further revealed that resonance can become chaotic, which has also been verified by experiments Sammarco et al. (1997b).

In view of the possible importance to offshore structures involving a periodic array of cylinders, we present here a nonlinear theory for subharmonic resonance of waves trapped by a vertical cylinder in a channel, that is mathematically equivalent to an infinite array of periodically spaced cylinders. The evolution equation for the amplitude of the trapped mode is found analytically to be of Landau-Stuart form. The main task of calculating the coupling coefficients is achieved by solving a number of scattering or radiation problems. By numerical solution of these problems, the effects of the geometry on the resonance characteristics are examined.

We consider a bottom-mounted circular cylinder of radius a' fixed at the center of a channel of width $2d'$ and depth h'. Let a Cartesian coordinate system be chosen such that the (x', y') plane coincides with the still free surface and z' points upward along the cylinder axis. A train of plane waves of amplitude A' arrives along the positive x' axis towards the cylinder. Let the fluid be incompressible and inviscid, and the flow be irrotational. Dimensionless variables are defined by using the spacing d' to scale the coordinates, cylinder radius and depth, and by using the typical amplitude A' to scale the free surface displacement. Similar to the excitation of trapped waves along a sloping beach (Minzoni & Whitham, 1977) or around the mobile gates of Venice (Sammarco et al., 1997a,b), it can be shown that a trapped wave of natural frequency ω and amplitude $O(1)$ can be excited nonlinearly by an incident and scattered wave system of order ϵ at the frequency 2ω. The long time scale of resonant growth is of the order $1/\omega\epsilon^2$.

Upon introducing the slow time $\tau = \epsilon^2 t$ and the expansions $\Phi{=}\Phi_0 + \epsilon\Phi_1 + \epsilon^2\Phi_2 + \cdots$, where Φ_n are functions of $(x, y, z; t, \tau)$, we obtain the governing perturbation equations for the first three orders, $n = 0, 1, 2$. Except for the boundary condition on the free surface,

$$\frac{\partial \Phi_n}{\partial z} + \frac{\partial^2 \Phi_n}{\partial t^2} = \mathscr{F}_n, \qquad \text{on} \quad z = 0, \tag{1}$$

all other conditions are homogeneous.

At the leading order, the boundary-value problem is homogeneous because $\mathscr{F}_1 = 0$. Let the eigenfunction be expressed as

$$\Phi_1 = \frac{B}{2}\varphi_1(x, y, z)\mathrm{e}^{-\mathrm{i}\omega t} + * = \frac{B}{2\,\mathrm{i}\omega}\frac{\cosh k(z+h)}{\cosh kh}\eta(x, y)\mathrm{e}^{-\mathrm{i}\omega t} + *, \tag{2}$$

where $B(\tau)$ is the complex amplitude, and asteriks denote complex conjugates. The spatial factors φ_1 and η satisfy the no-flux condition on channel walls and is antisymmetric about the channel middle plane $y = 0$. As shown in Callan et al. (1991), a trapped mode symmetric in x exists for all cylinder radius $0 < a < 1$ below the cutoff wavenumber $k < \pi/2$. Only in the small range of $0.81 < a < 1$, a second trapped mode antisymmetric in x with a different eigen-frequency exists (Evans & Porter, 1999). In the present study, we focus our attention only on the x-symmetric mode; the second mode can be treated similarly. For a vertical cylinder of circular cross-section, the eigen-wavenumber k depends only on the dimensionless radius a. Through the dispersion relation the eigen-frequency ω depends on the water depth in addition.

For any body geometry, the eigenvalue problem can be solved numerically by the hybrid-element technique (Chen & Mei, 1974; Mei, 1989). The basic idea is to employ two-dimensional finite elements only near the body where the geometry is complex, and analytical representations via eigenfunction expansions in the far field. The expansion coefficients in the far fields are found by a variational method along with the nodal unknowns in the near field.

At the order $O(\epsilon)$, the free surface condition is no longer homogeneous where $\mathscr{F}_2$ is given by

$$\mathscr{F}_2 = \frac{B^2}{2} q \mathrm{e}^{-2\mathrm{i}\omega t} + *, \quad \text{with} \quad q = \frac{1}{2\,\mathrm{i}\omega}\Big(2\eta_x^2 + 2\eta_y^2 + (3\omega^4 - k^2)\eta^2\Big). \tag{3}$$

Let the total solution be the sum of three parts $\Phi_2 = \Phi_I + \Phi_S + \Phi_Q$. where Φ_I and Φ_S are the incident and scattered wave potentials, respectively, and Φ_Q is the radiated wave forced on the free surface by quadratic interactions. The diffraction problem for frequency 2ω is standard in the linearized theory. The hybrid-element numerical scheme is used to compute the scattered waves near and away from the cylinder for a wide range of frequencies.

Since the forcing function on the free surface (3) contains only the second harmonic, we express

$$\Phi_Q = \frac{1}{2} B^2 \varphi_Q \mathrm{e}^{-2\mathrm{i}\omega t} + *, \tag{4}$$

so that φ_Q must satisfy the inhomogeneous free-surface condition:

$$\frac{\partial \varphi_Q}{\partial z} - 4\omega^2 \varphi_Q = q, \tag{5}$$

and the radiation condition that only outgoing waves exist at infinity.

Again the hybrid element method is used to solve for φ_Q. The far-field representation is easily found by separation of variables. In the near field, φ_Q will be approximated by three-dimensional finite elements with nodal unknowns. Along the interfaces $x = \pm L$ of the near and far fields, φ_Q and its x- derivative must be continuous. The combined problem for the entire channel is converted to a variational problem. Extremization leads to a linear matrix equation for the nodal unknowns and the expansion coefficients, and is then calculated numerically.

Finally, at the order $O(\epsilon^2)$, the free surface condition is inhomogeneous; forcing on the free surface contains first and third harmonics in time,

$$\mathscr{F}_3 = \mathscr{F}_{31} \mathrm{e}^{-\mathrm{i}\omega t} + \mathscr{F}_{33} \mathrm{e}^{-3\mathrm{i}\omega t} + *. \tag{6}$$

It can be shown that $\mathscr{F}_{31}$ decays exponentially in $|x|$ as the trapped wave. Let the third-order wave potential be expressed by

$$\Phi_3 = \varphi_{31}(x, y, z) \mathrm{e}^{-\mathrm{i}\omega t} + \varphi_{33}(x, y, z) \mathrm{e}^{-3\mathrm{i}\omega t} + *. \tag{7}$$

It can be shown that φ_{31} is governed by an inhomogeneous boundary-value problem whose homogeneous solution is φ_1. The solvability condition of the inhomogeneous problem for φ_{31} is found by applying Green's theorem to φ_1 and φ_{31} over the entire fluid domain. After using all the governing conditions, we find the evolution equation of the trapped wave amplitude $B(\tau)$

$$-\mathrm{i}\frac{\mathrm{d}B}{\mathrm{d}\tau} = c_\alpha B^2 B^* + c_\gamma A B^*, \tag{8}$$

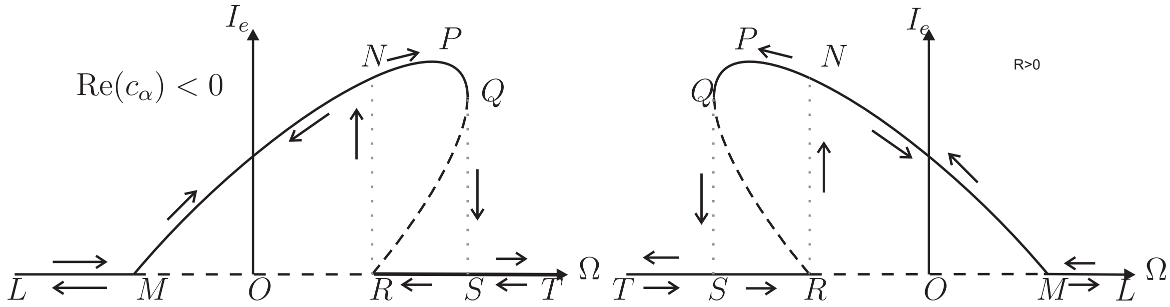

Figure 1: Bifurcation diagram relating the action I of equilibrium state and detuning frequency Ω for $\mathrm{Re}(c_\alpha) < 0$ (LEFT), and $\mathrm{Re}(c_\alpha) > 0$ (RIGHT). Solid line: stable branch; Dashed line: unstable branch.

which is of the Landau-Stuart form. If there is a small frequency detuning, the Landau-Stuart equation (8) becomes, after the transformation $B = \bar{B}\mathrm{e}^{-\mathrm{i}\Omega\tau}$,

$$-\mathrm{i}\frac{\mathrm{d}\bar{B}}{\mathrm{d}\tau} = c_\alpha|\bar{B}|^2\bar{B} + \Omega\bar{B} + c_\gamma\bar{B}^*. \tag{9}$$

The mathematical properties of this Landau-Stuart equation have been studied by Rockliff (1978) for edge waves on a beach. Let $\bar{B}$ be expressed in action-angle variables, i.e., $\bar{B} = \sqrt{I}e^{i\theta}$. For $\mathrm{Re}(c_\alpha) < 0$, the non-zero equilibrium state is shown as a right-leaning ellipse in the bifurcation diagram on the left of Figure (1), similar to a Duffing oscillator with a hard spring. Stable and unstable branches of the ellipse are shown in solid and dashed curves, respectively. Thus hysteresis is possible when the detuning frequency is varied. On the other hand when $\mathrm{Re}(c_\alpha) > 0$, the bifurcation diagram is a left-leaning ellipse, as shown on the right of Figure (1), similar to a soft spring. For physical understanding the dependences of the various coefficients on the cylinder-channel geometry have been examined from numerical solutions, and will be reported at the conference.

Since the mathematical problems of subharmonic resonance of trapped waves are essentially the same, be it an edge wave on a beach, a trapped mode around Venice gates, or a trapped mode around periodic cylinders, it is natural to anticipate that all trapped waves, whether around a stationary or mobile boundary, can be excited by the same mechanism.

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Discusser - D.V. Evans:

How is it possible to generate an anti-symmetric trapped mode ϕ_1 by a symmetric wave?

Reply:

In a linearized problem the response potential has the same spatial symmetry as the incident wave. In the nonlinear problem resonance is not forced directly by the incident wave, but by the *interaction* of the incident wave and the trapped wave, represented by the product AB^* in eq (8). Since the incident wave is symmetric in y while the trapped wave is antisymmetric, the forcing is antisymmetric. Note that nonlinear resonance is an instability problem; there has to be some trapped wave for it to be started. The incident-scattered wave system serves as a promoter. This is unlike linearized resonance which is an inhomogeneous problem mathematically and can be started from nothing.