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Wave Drift Force on a Floating Body in Two-Layer Fluids

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1. Introduction

Hydrodynamic studies of a body floating in two-layer fluids of finite depth have been made for the first-order radiation and diffraction problems (e.g. Ten & Kashiwagi (2004) and Kashiwagi et al. (2005)). However, to the author's knowledge, no study has been made so far on the second-order wave drift force in two-layer fluids.

The analyses for a two-layer fluid are rather complicated even for first-order problems. For example, in the diffraction problem, two different incident waves of surface-wave mode (with longer wavelength) and internal-wave mode (with shorter wavelength) must be considered for a prescribed frequency, and each incident wave will be scattered by a body into two different wave modes. Thus the energy of the incident wave may be transferred from one mode to another. Furthermore, when the body is oscillating in response to the incident wave, the body motion may change the reflected and transmitted waves. For this complicated wave field in a two-layer fluid, it is crucial to derive analytically the correct form of calculation formula for the wave drift force and to understand what are distinctive differences from the single-layer case.

2. Formulation

A 2-D floating body of general shape is considered, which intersects the interface between the upper and lower layers as a general case and oscillates sinusoidally in response to an incident wave with circular frequency ω. Analyses are performed with the coordinate system and notations shown in Fig. 1. The free surface, the interface, and the flat rigid bottom of water are located at $z = 0$, $z = h_1$, and $z = h\,(= h_1 + h_2)$, respectively.

Assuming a linear potential flow for both upper and lower fluids, the velocity potential is introduced and written in the form

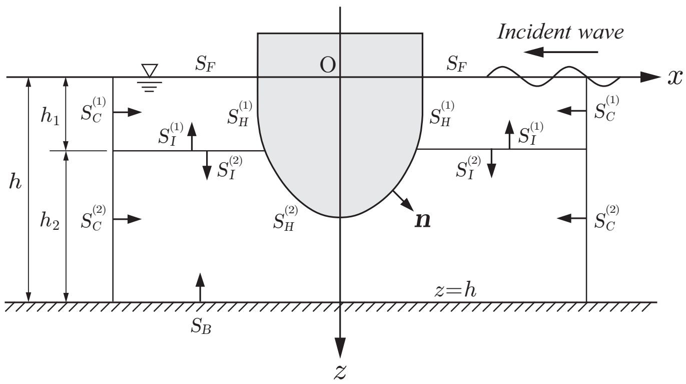

Fig. 1 Coordinate system and notations

$$\Phi^{(m)}(x,z,t) = \mathrm{Re}\Big[\,\phi^{(m)}(x,z)\,e^{i\omega t}\Big],\;\; m = 1,2, \tag{1}$$

$$\phi^{(m)}(x,z) = \sum_{p=1}^{2}\frac{gA_p}{i\omega}\left\{\phi_{Dp}^{(m)}(x,z) - K\sum_{j=1}^{3}\frac{X_j}{A_p}\,\phi_{Rj}^{(m)}(x,z)\right\} \equiv \sum_{p=1}^{2}\frac{gA_p}{i\omega}\,\varphi_p^{(m)}(x,z), \tag{2}$$

where
$$\phi_{Dp}^{(m)}(x,z) = \phi_{Ip}^{(m)}(x,z) + \phi_{Sp}^{(m)}(x,z), \tag{3}$$

and superscript (m) denotes the fluid layer, with $m = 1$ and 2 corresponding to the upper and lower fluid layers, respectively. $\phi_{Dp}^{(m)}$ denotes the diffraction potential which is the sum of the incident-wave potential $\phi_{Ip}^{(m)}$ and the scattering potential $\phi_{Sp}^{(m)}$. In two-layer fluids, there can be two different wave modes for a prescribed frequency; those modes are differentiated with subscript (p), and specifically $p = 1$ is referred to as the surface-wave mode and $p = 2$ as the internal-wave mode. $\phi_{Rj}^{(m)}$ denotes the radiation potential with unit velocity in the j-th direction ($j = 1$ for sway, $j = 2$ for heave, and $j = 3$ for roll), and the complex amplitude of the j-th mode of motion, X_j/A_p, must be determined by solving the motion equations of a body in waves.

The velocity potentials are governed by the Laplace equation and satisfy the linearized boundary conditions including the continuity condition of the vertical velocity and the pressure on the interface, which are well defined in related literatures (e.g. Ten & Kashiwagi (2004)).

The velocity potential of the incident wave (incoming from the positive x-axis) can be determined irrespective of the presence of a body, and the resultant expression for $\phi_{Ip}^{(m)}$ is written as

$$\phi_{Ip}^{(m)}(x,z) = Z^{(m)}(k_p;z)\, e^{ik_p x}\,, \tag{4}$$

where
$$Z^{(1)}(k;z) = \frac{k\,\mathrm{ch}kz - K\,\mathrm{sh}kz}{k}\,,\ Z^{(2)}(k;z) = \frac{K\,\mathrm{ch}kh_1 - k\,\mathrm{sh}kh_1}{k\,\mathrm{sh}kh_2}\,\mathrm{ch}k(z-h)\,. \tag{5}$$

Variable k in (4) and (5) is the wavenumber satisfying the dispersion relation for a two-layer fluid:

$$D(k) = K(k\,\mathrm{sh}kh - K\,\mathrm{ch}kh) + (1-\gamma)(K^2 - k^2)\,\mathrm{sh}kh_1\,\mathrm{sh}kh_2 = 0\,. \tag{6}$$

For brevity, the hyperbolic functions of $\cosh(x)$ and $\sinh(x)$ have been written as $\mathrm{ch}(x)$ and $\mathrm{sh}(x)$, respectively, $K = \omega^2/g$ with g being the gravitational acceleration, and $\gamma = \rho_1/\rho_2 \leq 1$ with ρ_m being the density of the upper ($m = 1$) and lower ($m = 2$) fluids.

3. Velocity Potentials at the Far Field

The diffraction and radiation potentials on the body surface are determined directly by the integral-equation method in terms of the Green function satisfying all homogeneous boundary conditions. More details for the derivation of the integral equation may be found in Ten & Kashiwagi (2004) and Kashiwagi et al. (2005).

Once the velocity potentials on the body surface are determined, it is straightforward to compute hydrodynamic forces and then to solve the motion equations of a body in each of the incident waves of surface-wave mode ($p = 1$) and internal-wave mode ($p = 2$).

The asymptotic expression of velocity potentials at $x \to \pm\infty$ can be obtained by neglecting evanescent wave terms in the Green function and substituting its result in the expression of the velocity potential derived with Green's theorem. Considering only the incident wave of k_p-wave mode, the resultant asymptotic expression for the velocity potential $\varphi_p^{(m)}$ defined in (2) can be expressed as follows:

$$\varphi_p^{(m)}(x,z) \sim Z^{(m)}(k_p;z)\, e^{ik_p x} + \sum_{q=1}^{2} R_{pq}\, Z^{(m)}(k_q;z)\, e^{-ik_q x} \quad \text{as } x \to +\infty\,, \tag{7}$$

$$\varphi_p^{(m)}(x,z) \sim \sum_{q=1}^{2} T_{pq}\, Z^{(m)}(k_q;z)\, e^{ik_q x} \quad \text{as } x \to -\infty\,, \tag{8}$$

where
$$R_{pq} = i\, H_p^{+}(k_q)\,,\ T_{pq} = \delta_{pq} + i\, H_p^{-}(k_q)\,, \tag{9}$$

$$H_p^{\pm}(k_q) = H_{Sp}^{\pm}(k_q) - K\sum_{j=1}^{2} \frac{X_j}{A_p}\, H_{Rj}^{\pm}(k_q)\,, \tag{10}$$

with δ_{pq} being Kronecker's delta.

R_{pq} and T_{pq} are the coefficients of reflected and transmitted waves, respectively, with wavenumber k_q when the incident wave is of the k_p-wave mode. As shown in (9) and (10), these can be computed by superposition of the Kochin functions in the diffraction and radiation problems.

4. Momentum and Energy Conservation Principles

The wave drift force in a two-layer fluid can be analyzed in the same way as that for a single-layer fluid on the basis of the momentum and energy conservation principles. First let us consider the momentum conservation in the x-axis in a two-layer fluid, which may be written in the form

$$\overline{\sum_{m=1}^{2} \int_{S^{(m)}} \left\{ p^{(m)} n_x + \rho_m \frac{\partial \Phi^{(m)}}{\partial x} \left(\frac{\partial \Phi^{(m)}}{\partial n} - U_n \right) \right\} ds} = 0\,, \tag{11}$$

where
$$S^{(1)} = S_H^{(1)} + S_C^{(1)} + S_I^{(1)} + S_F\,,\quad S^{(2)} = S_H^{(2)} + S_C^{(2)} + S_I^{(2)} + S_B\,, \tag{12}$$

$$p^{(m)} = -\rho_m \left\{ \frac{\partial \Phi^{(m)}}{\partial t} + \frac{1}{2} \nabla \Phi^{(m)} \cdot \nabla \Phi^{(m)} \right\} + p_S^{(m)}\,. \tag{13}$$

Here the overbar in (11) means time average over one period to be taken, U_n represents the normal velocity of the boundaries surrounding the fluid under consideration, and $p_S^{(m)}$ in (13) denotes the static pressure.

Considering the case where the incident wave is of the k_p-wave mode ($p = 1$ or 2) and retaining only quadratic terms in the velocity potential, the wave drift force acting in the negative x-axis may be written as follows:

$$
\begin{aligned}
F'_{Dp} &\equiv \frac{1}{\frac{1}{2}\rho_1 g A_p^2} \sum_{m=1}^{2} \overline{\int_{S_H^{(m)}} p^{(m)}\, n_x\, ds} \\
&= \frac{1}{2K}\Bigg[\int_0^{h_1} \left\{ \left| \frac{\partial \varphi_p^{(1)}}{\partial x} \right|^2 - \left| \frac{\partial \varphi_p^{(1)}}{\partial z} \right|^2 \right\} dz + \frac{1}{\gamma} \int_{h_1}^{h} \left\{ \left| \frac{\partial \varphi_p^{(2)}}{\partial x} \right|^2 - \left| \frac{\partial \varphi_p^{(2)}}{\partial z} \right|^2 \right\} dz \Bigg]_{-\infty}^{+\infty} \\
&\quad + \frac{1}{2}\Big[\left| \varphi_p^{(1)} \right|_{z=0}^2 \Big]_{-\infty}^{+\infty} + \frac{1-\gamma}{2\gamma K^2} \Bigg[\left| \frac{\partial \varphi_p^{(1)}}{\partial z} \right|_{z=h_1}^2 \Bigg]_{-\infty}^{+\infty}
\end{aligned}
\tag{14}
$$

where the square brackets with superscript $+\infty$ and subscript $-\infty$ means the difference between the quantities in the brackets evaluated at $x = +\infty$ and $x = -\infty$.

Equation (14) can be regarded as a reasonable extension from the single-layer result. However, this form is not convenient for analytical integration, because the derivatives with respect to z are included and thus the orthogonality properties of the eigenfunctions for a two-layer fluid can not directly be used. To surmount this inconvenience, further transformation is considered for the integrals containing the derivatives with respect to z by means of partial integrations and then the Laplace equation and the boundary conditions at $z = 0$, $z = h_1$, and $z = h$. The result can be of the following form:

$$
\begin{aligned}
\mathcal{I} &\equiv \int_0^{h_1} \left| \frac{\partial \varphi_p^{(1)}}{\partial z} \right|^2 dz + \frac{1}{\gamma} \int_{h_1}^{h} \left| \frac{\partial \varphi_p^{(2)}}{\partial z} \right|^2 dz \\
&= K \left\{ \left| \varphi_p^{(1)} \right|_{z=0}^2 + \frac{1-\gamma}{\gamma K^2} \left| \frac{\partial \varphi_p^{(1)}}{\partial z} \right|_{z=h_1}^2 \right\} + \int_0^{h_1} \frac{\partial^2 \varphi_p^{(1)}}{\partial x^2}\, \varphi_p^{(1)*}\, dz + \frac{1}{\gamma} \int_{h_1}^{h} \frac{\partial^2 \varphi_p^{(2)}}{\partial x^2}\, \varphi_p^{(2)*}\, dz\,,
\end{aligned}
\tag{15}
$$

where the asterisk means the complex conjugate.

Substituting this result in (14), it can be seen that the last two terms in (14) cancel out by the first two terms on the right-hand side of (15), and the result can be rewritten in the form

$$
F'_{Dp} = \frac{1}{2K} \Bigg[\int_0^{h_1} \left\{ \left| \frac{\partial \varphi_p^{(1)}}{\partial x} \right|^2 - \frac{\partial^2 \varphi_p^{(1)}}{\partial x^2}\, \varphi_p^{(1)*} \right\} dz + \frac{1}{\gamma} \int_{h_1}^{h} \left\{ \left| \frac{\partial \varphi_p^{(2)}}{\partial x} \right|^2 - \frac{\partial^2 \varphi_p^{(2)}}{\partial x^2}\, \varphi_p^{(2)*} \right\} dz \Bigg]_{-\infty}^{+\infty}
\tag{16}
$$

In a similar manner, starting from a basis equation due to the energy-conservation principle for the two-layer fluid and taking time average over one period, the result corresponding to (16) can be expressed as

$$
W'_p \equiv \frac{1}{\frac{1}{2}\rho_1 g A_p^2 \left(\frac{\omega}{K}\right)} \sum_{m=1}^{2} \overline{\int_{S_H^{(m)}} p^{(m)}\, U_n\, ds} = -\,\mathrm{Im} \Bigg[\int_0^{h_1} \frac{\partial \varphi_p^{(1)}}{\partial x}\, \varphi_p^{(1)*}\, dz + \frac{1}{\gamma} \int_{h_1}^{h} \frac{\partial \varphi_p^{(2)}}{\partial x}\, \varphi_p^{(2)*}\, dz \Bigg]_{-\infty}^{+\infty}
\tag{17}
$$

5. Calculation Formula of Wave Drift Force

In performing the integrals in (16) and (17) with respect to z, the orthogonality properties can be effectively used; those are written as follows. First let us define

$$
\mathcal{L}_{pq} \equiv \int_0^{h_1} Z^{(1)}(k_p; z) Z^{(1)}(k_q; z)\, dz + \frac{1}{\gamma} \int_{h_1}^{h} Z^{(2)}(k_p; z) Z^{(2)}(k_q; z)\, dz\,.
\tag{18}
$$

Then, in the same way as that in the Sturm-Liouville eigenvalue problem, it can be proven that $\mathcal{L}_{pq} = 0$ for $p \neq q$. For the case of $p = q$, $\mathcal{F}(k_p) \equiv 2k_p \mathcal{L}_{pp}$ can be written explicitly in the form

$$
\begin{aligned}
\mathcal{F}(k) &= \frac{K}{k} + kh \frac{(K \operatorname{ch} kh_1 - k \operatorname{sh} kh_1)^2}{\gamma\, k^2 \operatorname{sh}^2 kh_2} \\
&\quad + \frac{1-\gamma}{\gamma} \frac{h_1}{k} \left[\left(1 - \frac{k^2}{K^2} + \frac{1}{K h_1} \right) \big(K \operatorname{ch} kh_1 - k \operatorname{sh} kh_1 \big)^2 + \gamma \frac{(K^2 - k^2)^2}{K^2} \operatorname{sh}^2 kh_1 \right].
\end{aligned}
\tag{19}
$$

With these results, the integrals in (16) after substituting (7) and (8) for $\varphi_p^{(m)}$ can be performed analytically with relative ease, and the result can be expressed as

$$
F'_{Dp} = \frac{1}{2K} \left[k_p \left(1 + \left| R_{pp} \right|^2 - \left| T_{pp} \right|^2 \right) \mathcal{F}(k_p) + k_q \left(\left| R_{pq} \right|^2 - \left| T_{pq} \right|^2 \right) \mathcal{F}(k_q) \right].
\tag{20}
$$

In the same way, the result for the case of $W_p' = 0$ to be obtained from (17) can be expressed as

$$\left(1 - \left|R_{pp}\right|^2 - \left|T_{pp}\right|^2\right) \mathcal{F}(k_p) = \left(\left|R_{pq}\right|^2 + \left|T_{pq}\right|^2\right) \mathcal{F}(k_q). \tag{21}$$

In these expressions, convention of $p \neq q$ is adopted; that is, when $p = 1$ (the incident wave is of surface-wave mode) $q = 2$, and when $p = 2$ (the incident wave is of internal-wave mode) $q = 1$.

Combining (20) and (21), the final form of the formula for the wave drift force in two-layer fluids takes the following form:

$$F_{Dp}' = \frac{1}{K}\left[k_p \left|R_{pp}\right|^2 \mathcal{F}(k_p) + \left\{ \frac{k_p + k_q}{2}\left|R_{pq}\right|^2 + \frac{k_p - k_q}{2}\left|T_{pq}\right|^2 \right\} \mathcal{F}(k_q) \right]. \tag{22}$$

It should be noted here that there is a possibility of negative value of the wave drift force in a two-layer fluid for the case of $p = 1$. Because $k_2 > k_1$, the value of (22) can be negative if the value of $\left|T_{12}\right|$ (the transmitted wave with wavenumber k_2 in the incident wave of surface-wave mode) is relatively large. On the other hand, for the case of $p = 2$, the value of (22) is definitely positive.

6. Numerical Results

Numerical computations were implemented for a Lewis-form body with the half-breadth to draft ratio $H_0 = b/d = 0.833$ and the sectional area ratio $\sigma = A/Bd = 0.9$ (in real dimensions, the breadth $B = 0.2$ m and the draft $d = 0.12$ m). Owing to paucity of space, just one example among various results is shown in Fig. 2 for $h = 0.40$ m and $\gamma = 0.75$, in which only the heave motion is free to oscillate in response to the incident wave of surface-wave mode (left figure) and internal-wave mode (right figure). To see the effects of the interface, the vertical position of the interface was changed from $h_1 = 0.06$m to 0.20 m.

We can see that the drift force at $h_1 = 0.13$ m in the incident wave of surface-wave mode fluctuates in lower frequencies and becomes negative in a certain range of frequency, which is due to the effect of transmitted wave with internal-wave mode, as can be conjectured from (22).

In the incident wave of internal-wave mode, a remarkable change can be seen depending on whether a body intersects the interface. When the interface is deeper than the draft of a body, the drift force is negligibly small. However, once a body intersects the interface, the drift force becomes large and increases almost linearly with respect to Kb; for which we may envisage that the internal incident wave will be blocked by a body and almost all waves may be reflected. Namely the coefficient of R_{22} in the calculation formula (22) is largely different depending on whether the body intersects the interface.

References

Ten, I and Kashiwagi, M (2004). "Hydrodynamics of a body floating in a two-layer fluid of finite depth, Part-1: radiation problem", *J Marine Science & Technology*, Vol. 9, No. 3, pp. 127–141.

Kashiwagi, M, Ten, I and Yasunaga, M (2005). "Hydrodynamics of a body floating in a two-layer fluid of finite depth, Part-2: diffraction problem and wave-induced motions", *J Marine Science & Technology*, accepted and in press.

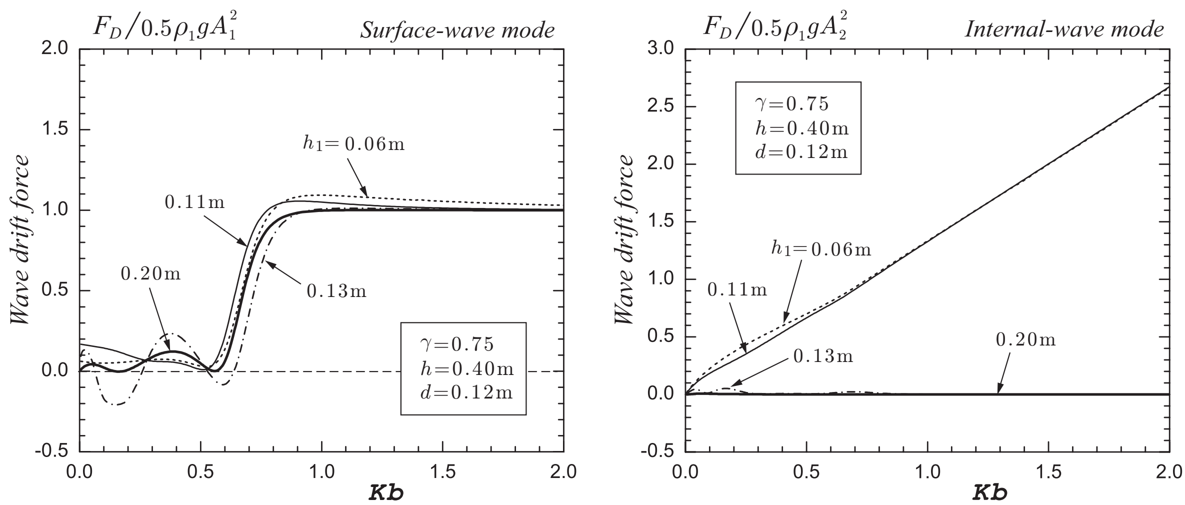

Fig. 2 Wave drift force on a Lewis-form body ($H_0 = 0.833$, $\sigma = 0.9$) in two-layer fluids. Effect of the interface position for the case where only the heave motion is free to oscillate.

Kashiwagi, M.
'Wave drift force on a floating body in two-layer fluids'

Discusser - C. C. Mei:

For 1-layered fluid, the method of radiation stress gives a convenient and physical way of delivering the drift force in terms of the reflection coefficient in the far field. Have you tried it?

Reply:

No, I have not. I suppose the method of radiation stress may give the same result for the drift force in two-layer fluids. One of the points in the present paper is to show a mathematical transformation which enables it easier to apply the orthogonality relations to the equations based on the momentum and energy conservation principles.

Discusser - T. Miloh:

The range of density ratios which you used $0 \leq \gamma < 1$ is a bit unrealistic. For real sea conditions $\gamma \sim 0.99$ and the interface (baroclinic mode) is the dominant mode.

I wonder if there are any limitations for applying your approach to a wave maker piercing the interface in the baroclinic approximation. In this case finding the intersection point between the interface and the body is not a trivial task and can not be presumed apriori.

Your discrete set of eigenfunctions can be applied only for the case of two-layers of finite depth. If the lower layer is infinitely deep a continuous (integral) set of eigenvalues should be used instead.

Reply:

Your saying is true in a real sea. In our laboratory experiment, the water and iso-paraffin oil were used, which gave the density ratio $\gamma \simeq 0.75$. A situation of two-layer fluid with large difference in the fluid density might be realized in a shallow channel with a large amount of muddy water on the bottom.

Numerical computations in the present paper are based on the linear theory and the integral equation method using the Green function. Therefore the intersection point can be clearly defined, and there is no numerical difficulty if the field and source points are not exactly at the intersection point. Numerical solution of the integral equation is obtained with the constant-panel and collocation method.

In the far-field method, it suffices to consider the progressive wave terms and thus the number of wavenumbers (eigenvalues) to be considered is only two. For the case of infinitely deep lower layer, the expression of the wavenumbers and eigenfunctions becomes simpler and analytical integrations with respect to z may be performed with relative ease without explicitly using the orthogonality relations.