Expansion formula in wave structure interaction problems-revisited

D. Karmakar and T.Sahoo
Department of Ocean Engineering and Naval Architecture
Indian Institute of Technology, Kharagpur – 721302, India
tsahoo@naval.iitkgp.ernet.in, dev_naval@yahoo.com

1. Introduction

In recent decades, there is growing interest in analyzing the dynamic response of floating elastic structures with ocean waves which plays a significant role in marine technology and in cold region engineering. These types of problems lead to a special class of boundary value problems associated with Laplace equation having higher order boundary conditions. Recently, Manam et al. (2006) developed expansion formulae for such type of wave structure interaction problems based on the direct application of Fourier analysis and Green's integral theorem In the present paper, the expansion formula in quarter plane for a more general type of boundary value problem as in Manam et al. (2006) is obtained. The detail derivation of the expansion formula is demonstrated by a different method in a particular case by analyzing the boundary value problem associated with the scattering of surface water waves by a discontinuity in a floating elastic plate.

2. General boundary value problem and its expansion formula

In the present paper, we consider the boundary value problem associated with a Laplace equation in two-dimensional Cartesian co-ordinate system, which arises in the broad area of fluid structure interaction with the fluid assumed to be inviscid and incompressible and the flow is assumed to be irrotational and simple harmonic in time with angular frequency ω. Thus, there exists a velocity potential $\Phi(x, y, t)$ of the form $\Phi(x, y, t) = \mathrm{Re}[\phi(x, y)e^{-i\omega t}]$ which satisfies the Laplace equation in the space variables. The fluid is assumed to occupy region $0 < x < \infty, 0 < y < \infty$. On the structural boundary, the velocity potential $\phi(x, y)$ satisfies the boundary condition of the form

$$L\left(\partial_x\right)\phi_y + M\left(\partial_x\right)\phi = 0 \text{ on } y = 0, 0 < x < \infty \tag{1}$$

where L and M are the linear differential operators of the form $L\left(\partial_x\right) = \sum_{n=0}^{n_0} c_n \partial_{x^{2n}}$, $M\left(\partial_x\right) = \sum_{n=0}^{m_0} d_n \partial_{x^{2n}}$ with c_n's and d_n's are the known constants. Further, the far field radiation condition is of the form

$$\phi\left(x, y\right) \sim \text{multiple of } e^{ik_0 x - k_0 y} \text{ as } x \to \infty \tag{2}$$

where k_0 satisfies the relation $\sum_{n=0}^{m_0} (-1)^n d_n k_0^{2n} = k_0 \left\{ \sum_{n=0}^{n_0} (-1)^n c_n k_0^{2n} \right\}$. (3)

Finally, the bottom boundary condition is given by

$$\phi,\ \nabla\phi \to 0, \quad \text{as } y \to \infty. \tag{4}$$

The expansion formula for the velocity potential $\phi\left(x, y\right)$ satisfying the conditions (1), (2) and (4) with $M\left(\partial_x\right) = d_0$ (a constant) is derived by Manam et al. (2006) by applying the Fourier sine transform to the vertical boundary in which the unknown coefficients are obtained by the use of a newly defined mode coupling relation. On the other hand, in the present paper, Fourier sine transform is applied on the horizontal boundary to convert the BVP to a Sturm-Liouville type BVP associated with non-homogeneous ordinary differential equation (ODE) in the transformed variable. The solution of the ODE in the transformed variable is obtained by the Green's function technique. Finally, inverting the transformed functions and applying the regularity criterion of the transformed functions, the required expansion formula is derived. The same approach is being extended to derive the expansion formula in a semi-infinite strip and details are deferred here. Here, without going to the detail derivation, we mentioned the general form of the expansion formula in terms of a Theorem.

Theorem: The velocity potential $\phi(x,y)$ satisfying the governing equation along with the boundary conditions (1) and (2) in case of infinite depth is given by

$$\phi(x,y)=\sum_{n=0,I}^{2n_0}A_n e^{ik_nx-k_ny}+\frac{2}{\pi}\int_0^\infty \frac{L(\xi,y)A(\xi)e^{-\xi x}d\xi}{\xi^2\left(\sum_{n=0}^{n_0}c_n\xi^{2n}\right)^2+\left(\sum_{n=0}^{m_0}d_n\xi^{2n}\right)^2},\ \ n=0,I,II,..,2n_0 \tag{5}$$

$$A_ne^{ik_nx}=\frac{\left[\left\{\sum_{j=1}^{n_0}(-1)^{j+1}c_j\sum_{k=1}^{j}k_n^{2(j-k)}\phi^{(2k-1)}(x,0)\right\}k_n+\left\{\sum_{j=1}^{m_0}(-1)^{j+1}d_j\sum_{k=1}^{j}k_n^{2(j-k)}\phi^{(2k-2)}(x,0)\right\}k_n\right]}{C_n\sum_{j=0}^{m_0}(-1)^j d_jk_n^{2j}}+\frac{\int_0^\infty e^{-k_nt}\phi(x,t)\,dt}{C_n}$$

$$C_n=\frac{\sum_{j=0}^{n_0}(-1)^j(2j+1)c_jk_n^{2j}+\sum_{j=0}^{m_0}(-1)^{j+1}2jd_jk_n^{2j-1}}{2\sum_{j=0}^{m_0}(-1)^jd_jk_n^{2j}},\ \ L(\xi,y)=\left(\sum_{n=0}^{n_0}c_n\xi^{2n}\right)\xi\cos\xi y-\left(\sum_{n=0}^{m_0}d_n\xi^{2n}\right)\sin\xi y$$

$$A(\xi)e^{-\xi x}=\int_0^\infty\phi(x,t)L(\xi,t)dt+\left\{\sum_{j=1}^{n_0}c_j\sum_{k=1}^{j}\xi^{2(j-k)}\phi^{(2k-1)}(x,0)\right\}\xi+\left\{\sum_{j=1}^{m_0}d_j\sum_{k=1}^{j}\xi^{2(j-k)}\phi^{(2k-2)}(x,0)\right\}\xi$$

where, $k_n,\ \ n=I,..,2n_0$ are the complex roots of the dispersion relation. The detail proof of the theorem is deferred here and will be presented in the workshop.

3. Scattering of surface water waves by a discontinuity in a floating elastic plate

In the present paper, the scattering of surface water waves by a discontinuity at origin in an infinitely extended floating elastic plate is analyzed by considering the Timoshenko-Mindlin equation as the plate equation instead of the Euler-Bernoulli beam equation to include the plate thickness. It may be noted that this point of discontinuity is referred to as a crack in case of a floating ice sheet (as in Evans and Porter (2003)). Without going into the detail derivation, we will use the convention as used in the paper of Balmforth and Craster (1999) which yields the plate covered free surface condition as in (1) in two-dimensions with $n_0=2$ and $m_0=1$ and is given by

$$\left(c_0+c_1\partial_{xx}+c_2\partial_{xxxx}\right)\phi_y+\left(d_0+d_1\partial_{xx}\right)\phi=0,\ \ \text{on } y=0,\ -\infty<x<0, 0<x<\infty \tag{6}$$

where $c_0=\left\{m^2\omega^4\left(IS/B\right)+\rho_wg-m\omega^2-im\pi\omega\right\}$, $c_1=m\omega^2\left(S+I\right)$, $c_2=B$, $d_0=\rho_w\omega^2\left\{1-\omega^2m\left(IS/B\right)\right\}$, $d_1=-\rho_w\omega^2S$, $I=d^2/12$ is rotary inertia, $B=Ed^3/\left\{12(1-\nu^2)\right\}$ is the plate rigidity, $S=12B/\pi^2Gd$ is the shear deformation of the plate, $G=E/2\left(1+\nu\right)$ is the shear modulus of elastic material, E is the Young's modulus, π is the second effect of damping, $m=\rho_pd$ is mass per unit area, ρ_p is the density of the plate, ρ_w is the density of water, g is the acceleration due to gravity and d is the draft of the elastic plate. The far field condition is of the form

$$\phi(x,y)\sim\begin{cases}\left(e^{-ik_0x}+R_0\,e^{ik_0x}\right)e^{-k_0y} & \text{as}\quad x\to\infty,\\ T_0e^{-ik_0x}e^{-k_0y} & \text{as}\quad x\to-\infty,\end{cases} \tag{7}$$

where R_0 and T_0 are the unknown reflection and transmission coefficients to be determined as a part of the solution procedure. Assuming that the plates are having a line discontinuity at the origin with free edge behavior, the vanishing of shear force and the bending moment at this end yields

$$\phi_{yyy}\left(0\pm,0\right)=0,\ \phi_{xyyy}\left(0\pm,0\right)=N\phi_{xy}\left(0\pm,0\right)\ \text{with}\ N=c_1/c_2 \tag{8}$$

In addition, across the boundary between the two plate covered region, the continuity of velocity and pressure near the point of discontinuity yields

$$\phi_x(x+,y)=\phi_x(x-,y) \text{ and } \phi(x+,y)=\phi(x-,y) \text{ at } x=0, 0<y<\infty. \tag{9}$$

In order to solve this BVP defined in the half plane, it is reduced to two quarter plane boundary value problems in $\varphi(x,y)$ and $\Upsilon(x,y)$ which are defined as

$$\varphi(x,y)=\phi(x,y)-\phi(-x,y) \text{ and } \Upsilon(x,y)=\phi(x,y)+\phi(-x,y). \tag{10}$$

Apart from satisfying conditions of the form (1), (2) and (3), the reduced potentials $\varphi(x,y)$ and $\Upsilon(x,y)$ satisfy the boundary condition (because of Eq. (9))

$$\Upsilon_x(x,y)=0 \text{ and } \varphi(x,y)=0 \text{ at } x=0, \quad 0<y<\infty. \tag{11}$$

In order to find the potential function $\varphi(x,y)$, we put

$$\varphi(x,y)=e^{-ik_0x-k_0y}+A_0e^{ik_0x-k_0y}+\psi(x,y) \tag{12}$$

and taking the Fourier sine transform of $\psi(x,y)$ as given by $\hat{\psi}_s(\xi,y)=\int_0^\infty \psi(x,y)\sin\xi x dx$, the boundary value problem in terms of $\varphi(x,y)$ is converted into a boundary value problem associated with an ordinary differential equation in terms of $\hat{\psi}_s(\xi,y)$ whose solution is given by

$$\hat{\psi}_s(\xi,y)=\frac{S(\xi,y)}{\xi H(\xi)}\left\{\frac{1+A_0}{(k_0+\xi)}+\frac{\hat{f}(\xi)}{(d_0-d_1\xi^2)}\right\}+\frac{1}{\xi}\int_0^y \sinh\xi(t-y)g(\xi,t)dt+\frac{\hat{f}(\xi)}{(d_0-d_1\xi^2)} \tag{13}$$

with $S(\xi,y)=\xi(c_0-c_1\xi^2+c_2\xi^4)\cosh\xi y-(d_0-d_1\xi^2)\sinh\xi y$, $H(\xi)=\xi(c_0-c_1\xi^2+c_2\xi^4)-(d_0-d_1\xi^2)$ and $g(\xi,t)=\xi(1+A_0)e^{-k_0t}+\xi^2\hat{f}(\xi)/(d_0-d_1\xi^2)$. $H(\xi)$ satisfies the relation in (4) for $\xi=k_0$ which shows that the transformed function $\hat{\psi}_s(\xi,y)$ has a singularity at $\xi=k_0$ on the positive real axis. This gives rise to the requirement that

$$\lim_{\xi\to k_0}(\xi-k_0)\hat{\psi}_s(\xi,y)=0 \tag{14}$$

which yields

$$A_0=\frac{2k_0\left\{\alpha_1(c_1-c_2k_0^2)+\alpha_0d_1\right\}}{(5c_2k_0^4-3c_1k_0^2+2d_1k_0+c_0)}-1 \tag{15}$$

with $\alpha_0=\varphi(0+,0), \alpha_1=\varphi_y(0+,0)$. Taking the inverse Fourier sine transform of $\hat{\psi}_S(\xi,y)$ and using the Cauchy residue theorem of complex function theory, $\psi(x,y)$ is obtained as

$$\psi(x,y)=A_Ie^{ik_Ix-k_Iy}+A_{II}e^{-ik_{II}x-k_{II}y}+\frac{2}{\pi}\int_0^\infty \frac{L(\xi,y)A(\xi)e^{-\xi x}d\xi}{\Delta(\xi)} \tag{16}$$

where

$L(\xi,y)=\xi(c_0+c_1\xi^2+c_2\xi^4)\cos\xi y-(d_0+d_1\xi^2)\sin\xi y$, $\Delta(\xi)=\xi^2(c_0+c_1\xi^2+c_2\xi^4)^2+(d_0+d_1\xi^2)^2$,

$A_n=\dfrac{2k_n\left\{\alpha_1(c_1-c_2k_n^2)+\alpha_0d_1\right\}}{(5c_2k_n^4-3c_1k_n^2+2d_1k_n+c_0)}$ $(n=I,II)$, $A(\xi)=\dfrac{\alpha_1\left\{c_2L'''(\xi,0)-c_1L'(\xi,0)\right\}}{(d_0+d_1\xi^2)}+\dfrac{d_1\alpha_0L(\xi,0)}{\xi(c_0+c_1\xi^2+c_2\xi^4)}$.

Substituting for A_0 and $\psi(x,y)$ from (16) in relation (12), the expansion formulae is obtained as

$$\varphi(x,y)=e^{-ik_0x-k_0y}+A_0e^{ik_0x-k_0y}+A_Ie^{ik_Ix-k_Iy}+A_{II}e^{-ik_{II}x-k_{II}y}+\frac{2}{\pi}\int_0^\infty \frac{L(\xi,y)A(\xi)e^{-\xi x}d\xi}{\Delta(\xi)}. \tag{17}$$

Now, applying the plate edge conditions as in Evans and Porter (2003), the unknowns α_0 and α_1 are obtained as $\alpha_0 = J_1\alpha_1/J_0$ and $\alpha_1 = 2ik_0^2\left(N-k_0^2\right)\big/\beta_1$

with $J_0 = \sum_{n=0}^{II}\frac{2d_1k_n^4}{\left(5c_2k_n^4-3c_1k_n^2+2d_1k_n+c_0\right)} - \frac{2}{\pi}\int_0^\infty \frac{d_1L'''(\xi,0)L(\xi,0)d\xi}{\Delta(\xi)\xi\left(c_0+c_1\xi^2+c_2\xi^4\right)}$,

$$J_1 = -\sum_{n=0}^{II}\frac{2k_n^4\left(c_1-c_2k_n^2\right)}{\left(5c_2k_n^4-3c_1k_n^2+2d_1k_n+c_0\right)} + \frac{2}{\pi}\int_0^\infty \frac{L'''(\xi,0)\left\{c_2L'''(\xi,0)-c_1L'(\xi,0)\right\}d\xi}{\Delta(\xi)\left(d_0+d_1\xi^2\right)}$$ and

$$\beta_1 = \sum_{n=0}^{II}\frac{2ik_n^3\left(N-k_n^2\right)\left\{\left(c_1-c_2k_n^2\right)+\left(J_1d_1/J_0\right)\right\}\varepsilon_n}{\left(5c_2k_n^4-3c_1k_n^2+2d_1k_n+c_0\right)} + \frac{2}{\pi}\int_0^\infty \frac{\xi\left\{NL'(\xi,0)-L'''(\xi,0)\right\}A(\xi)d\xi}{\alpha_1\Delta(\xi)}.$$

Considering $\varphi(x,y) = \Upsilon_x(x,y)$ and proceeding in the similar way as done earlier we obtain

$$\Upsilon(x,y) = e^{-ik_0x-k_0y} + B_0e^{ik_0x-k_0y} + B_Ie^{ik_Ix-k_Iy} + B_{II}e^{-ik_{II}x-k_{II}y} + \frac{2}{\pi}\int_0^\infty \frac{L(\xi,y)B(\xi)e^{-\xi x}d\xi}{\Delta(\xi)}, \qquad (18)$$

where $B_n = \delta_n - \frac{2i\left\{\alpha_2\left(c_1-Nc_2-k_n^2c_2\right)+\alpha_3d_1\right\}\varepsilon_n}{\left(5c_2k_n^4-3c_1k_n^2+2d_1k_n+c_0\right)}$, $\varepsilon_n = \begin{cases}1 & \text{for } n=0,I,\\ -1 & \text{for } n=II,\end{cases}$ $\delta_n = \begin{cases}1 & \text{for } n=0,\\ 0 & \text{for } n=I,II\end{cases}$ and

$B(\xi) = \frac{\alpha_2\left\{c_1L'(\xi,0)-c_2NL'(\xi,0)-c_2L'''(\xi,0)\right\}}{\xi\left(d_0+d_1\xi^2\right)} - \frac{\alpha_3d_1L(\xi,0)}{\xi^2\left(c_0+c_1\xi^2+c_2\xi^4\right)}$ with $\alpha_2 = \Upsilon_{xy}(0+,0)$, $\alpha_3 = \Upsilon_x(0+,0)$.

Proceeding in a similar manner as in case of α_0 and α_1, here α_2 and α_3 are obtained as $\alpha_2 = 2k_0^3\big/\beta_2$ and

$\alpha_3 = J_2\alpha_2/J_3$ with $J_3 = -\sum_{n=0}^{II}\frac{2k_n^2\left(N-k_n^2\right)d_1}{\left(5c_2k_n^4-3c_1k_n^2+2d_1k_n+c_0\right)} + \frac{2}{\pi}\int_0^\infty \frac{\xi^2d_1\left\{L'''(\xi,0)-NL'(\xi,0)\right\}L(\xi,0)d\xi}{\xi\left(c_0+c_1\xi^2+c_2\xi^4\right)\Delta(\xi)}$,

$$J_2 = \sum_{n=0}^{II}\frac{2k_n^2\left(N-k_n^2\right)\left(c_1-Nc_2-k_n^2c_2\right)}{\left(5c_2k_n^4-3c_1k_n^2+2d_1k_n+c_0\right)} + \frac{2}{\pi}\int_0^\infty \frac{\xi^2\left\{NL'(\xi,0)-L'''(\xi,0)\right\}\left\{\left(c_1-c_2N\right)L'(\xi,0)-c_2L'''(\xi,0)\right\}d\xi}{\left(d_0+d_1\xi^2\right)\Delta(\xi)},$$

and $\beta_2 = \sum_{n=0}^{II}\frac{2ik_n^3\left\{\left(c_1-Nc_2-k_n^2c_2\right)+\left(J_2d_1/J_3\right)\right\}\varepsilon_n}{\left(5c_2k_n^4-3c_1k_n^2+2d_1k_n+c_0\right)} + \frac{2}{\pi}\int_0^\infty \frac{B(\xi)L'''(\xi,0)d\xi}{\alpha_2\Delta(\xi)}$.

Once the constants A_0 and B_0 are determined, the reflection and transmission coefficients are evaluated from the relation $K_r = |R_0| = \left|\left(B_0+A_0\right)/2\right|$ and $K_t = |T_0| = \left|\left(B_0-A_0\right)/2\right|$. Numerical results with realistic dimensions as computed using the present approach will be presented at the workshop.

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Discusser - D.V. Evans:

In your general theory the constants c_i, $(i = 0, 1, \ldots)$ were assumed to be real. But in the example of the Mindlin plate c_0 appears to be complex.

Reply:

You are right. I have to check the formulation base on the Mindlin equation.

Discusser - M. H. Meylan:

Do you think you could derive a formula for the Green's function for a Mindlin plate (analogous to the Green's function for a thin plate given in Evans & Porter 21IWWWFB p46, eq 6)?

Reply:

Yes a Green's function can be derived following a similar approach.

Discusser - R. Porter:

The curves you present of the reflection coefficient appear to be self-similar. ie there is a simple scaling by which all curves will collapse to a single curve. Can a non-dimensional analysis of the governing equations resolve this?

Reply:

This is an interesting observation. We will look into the problem again.