Hydro-elastic Wagner impact using variational inequalities

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Introduction

A simple model of water impact has been introduced by Wagner [6]. This model is still a main tool in analysis of loads, which act on both rigid and elastic bodies entering water. The two-dimensional and axisymmetrical rigid body cases have been extensively studied. Wagner model has been extended to solve the three-dimensional problems of rigid body impact under the Wagner assumptions in [3] and [1].
However, most destructive fluid-structure impacts cannot be accurately modelled if the hydro-elastic coupling phenomenon is not considered. 2D modal and beam finite element methods to study the hydro-elastic impact have been developed in [4] and [2]. The coupled problem of fluid-structure impact for axisymmetrical cases was studied by modal method in [5]. However the methods developed for 2D cases cannot be directly applied to 3D impact problems. A main difficulty in treating 3D elastic structure impact is due to unknown in advance geometry of the contact region between the entering deformable body and liquid.

We present a method to solve the hydro-elastic 3D Wagner problem for a linear elastic structure. This method considers the fluid and the structure problems as two separated problems, which are solved alternatively until convergence with proper regularization. Within this method the original problem is reduced to a fixed-point problem at each time step. Numerical stability and convergence of the process are studied and justified with the help of a simplified one-degree of freedom model.
A preliminary 2D code was developed. The obtained results are compared with semi-analytical solution [4]; good agreement is found.

1 The coupled problem

The rigid body problem

A three-dimensional ideal irrotational incompressible flow is considered. The liquid is initially at rest and occupies the lower half space $\Omega = \{(\mathrm{x},\mathrm{y},\mathrm{z}) \in \mathbb{R}^2 \times \mathbb{R}^-\} = \{\mathrm{z} \leqslant 0\}$. A blunt rigid body starts to enter the liquid at time instant $t = 0$.

A variational inequality approach has been first introduced in [3]. A reader may refer to [3] and [1] for further details about the variational inequality formulation of the rigid body impact problem.

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Within the Wagner approach and using the displacement potential ϕ, defined as the integral of the velocity potential φ with respect to time,

$$\phi(\mathrm{x},\mathrm{y},\mathrm{z},t) = \int_0^t \varphi(\mathrm{x},\mathrm{y},\mathrm{z},\tau)\, d\tau, \tag{1}$$

the boundary value problem (BVP) with respect to the potential ϕ has the form

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega = \{\mathrm{z} \leqslant 0\} \\ \phi = 0 & \text{on the free surface} \\ \dfrac{\partial\phi}{\partial \mathrm{z}} = f(\mathrm{x},\mathrm{y}) - h(t) & \text{on the wet surface} \\ \phi \to 0 & \text{when } \mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2 \to \infty \end{cases} \tag{2}$$

where $f(x,y)$ is the body shape function and $h(t)$ is the penetration depth, $h(0) = 0$, $f(0,0) = 0$ and $f(x,y) \geqslant 0$. Free and wet surfaces are unknown and have to be determined as a part of the solution.
Using the displacement potential, two inequalities can be obtained (see [1] for details)

$$\phi \leqslant 0 \text{ on } \{z = 0\} \tag{3}$$

$$\frac{\partial\phi}{\partial z} \leqslant f(\mathrm{x},\mathrm{y}) - h(t) \text{ on } \{z = 0\} \tag{4}$$

System (2)-(4) can be reduced to a variational inequality:

$$a(\phi, v-\phi) \geqslant l(v-\phi)\ \forall v \in K, \tag{5}$$

where $K \subset W^1(\Omega)$ is the convex set of elements of $W^1(\Omega)$ which are negative or zero on $\{\mathrm{z} = 0\}$, and $W^1(\Omega)$ is defined as:

$$W^1(\Omega) = \left\{ v\ ;\ \frac{v}{\sqrt{1+|\vec{\mathrm{x}}|^2}}, \frac{\partial v}{\partial \mathrm{x}}, \frac{\partial v}{\partial \mathrm{y}}, \frac{\partial v}{\partial \mathrm{z}} \in L^2(\Omega) \right\}. \tag{6}$$

The bilinear form $a(\cdot,\cdot)$ is derived from the Laplacian operator and the linear form $l(\cdot)$ is given as

$$l(v) = \iint_{\{\mathrm{z}=0\}} (f(x,y) - h(t))\, v\, d\mathrm{x}\, d\mathrm{y}. \tag{7}$$

The variational inequality (5) can be reduced to a well-posed constrained minimization problem:

$$\min_{\phi \in K} \left(\frac{1}{2} a(\phi,\phi) - l(\phi) \right). \tag{8}$$

A method to solve this problem was described and tested in [1].

The hydro-elastic problem

For the problem of flexible structure impact the body boundary condition in (2) takes the form $\frac{\partial \phi}{\partial \mathrm{z}} = f(\mathrm{x},\mathrm{y}) + w(\mathrm{x},\mathrm{y},t) - h(t)$, where $w(\mathrm{x},\mathrm{y},t)$ is the normal deflection of the structure (see figure 1). Inequality (4) is written now as $\frac{\partial \phi}{\partial \mathrm{z}} \leqslant f(\mathrm{x},\mathrm{y}) + w(\mathrm{x},\mathrm{y},t) - h(t)$ along the liquid surface $z = 0$.

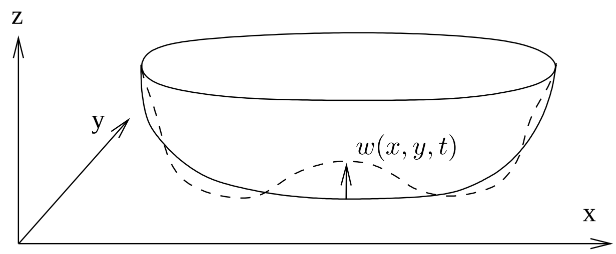

Figure 1: Water impact for a flexible structure

It is important to notice that the time t plays a role of parameter in (2) - (4). Therefore, even in the case of elastic structure impact, the displacement potential can be obtained at each time instant independently of the process history, if the structure deflection is known. This is an important peculiarity of the Wagner problem. If the deflection is prescribed, then the displacement potential can be obtained as the solution of the constrained minimization problem (8), where now the linear form (7) must be presented as

$$l(v) = \iint_{\{\mathrm{z}=0\}} (f(x,y) + w(x,y,t) - h(t))\; v(x,y)\; d\mathrm{x}\; dy. \quad (9)$$

By using either finite-element method or 3D normal mode method or any other method to discretize the elastic deflection of the penetrating structure, we represent the deflection with the help of a vector $\mathbf{W}(t)$. For example, in the modal method

$$w(x,y,t) = \sum_{n=1}^{\infty} W_n(t)\psi_n(x,y),$$

where $\psi_n(x,y)$ are the shape functions of the structure vibration. By using this discretization, one can present the solution of problem (8), (9) as

$$\phi = \Phi(\mathbf{W}), \quad (10)$$

where Φ is a nonlinear operator. Correspondingly, equation of the structure dynamics after discretization takes the form

$$\mathbb{M}(\frac{\partial^2 \mathbf{W}}{\partial t^2}) + \mathbb{K}(\mathbf{W}) = -\rho\mathbb{R}(\frac{\partial^2 \phi}{\partial t^2}), \quad (11)$$

where $\mathbb{M}$ is the structural mass matrix, $\mathbb{K}$ is the stiffness matrix, ρ is the liquid density and $\mathbb{R}$ is a linear operator, the form of which is dependent on the way of the elastic deflection discretization. In order to obtain equation (11), the Bernoulli equation $p = -\rho\phi_{tt}$ has been used, where $p(x,y,z,t)$ is the hydrodynamic pressure. Note that system (10) - (11) is coupled. This is, the potential ϕ and the deflection vector $\mathbf{W}(t)$ should be obtained at the same time.

Time discretization

Implicit scheme is used to discretize equation (11) in time:

$$\mathbb{M}\Big(\frac{\mathbf{W}^{n+1} - 2\mathbf{W}^n + \mathbf{W}^{n-1}}{\Delta t^2}\Big) + \mathbb{K}(\frac{\mathbf{W}^{n+1} + \mathbf{W}^{n-1}}{2}) = -\rho\mathbb{R}(\frac{\phi^{n+1} - 2\phi^n + \phi^{n-1}}{\Delta t^2}) + O(\Delta t^3), \quad (12)$$

where $\phi^n = \Phi(\mathbf{W}^n)$. This scheme keeps the problem coupled after the discretization and it is unconditionally stable in time for uncoupled problems, when ϕ is independent of $\mathbf{W}$. Searching for the unknown $\mathbf{W}^{n+1}$ and ϕ^{n+1}, one assumes that the solution is already known at the previous time steps. With $\mathbf{G}^n = \mathbb{M}(2\mathbf{W}^n - \mathbf{W}^{n-1}) + \rho\mathbb{R}(2\phi^n - \phi^{n-1}) - \frac{1}{2}\Delta t^2\mathbb{K}(\mathbf{W}^{n-1})$ equation (12) is written as:

$$(\mathbb{M} + \frac{1}{2}\Delta t^2\mathbb{K})(\mathbf{W}^{n+1}) = \mathbf{G}^n - \rho\mathbb{R}(\phi^{n+1}). \quad (13)$$

Note that equation (13) does not require calculations of the hydrodynamic loads acting on the structure but only the displacement potential at each time instant. If it is possible to distinguish the linear part of the operator Φ and combine this linear part with the left-hand side of equation (13), then we arrive at a stable and efficient numerical algorithm. However, if one intends to use a commercial code for the structural analysis, which does not provide matrices $\mathbb{M}$ and $\mathbb{K}$ as output, then this way is not practical and one is forced to deal with uncoupled problem.

2 Numerical coupling scheme

The final aim of this study is to couple the Wagner model for hydrodynamic loads with a closed source professional finite element package. One of the limitations of these packages is that one does not have access to the core of the software. This is why it is not possible, or it would be very tough, to solve the structure and fluid problems simultaneously.

It is suggested to solve the coupled problem (13) by iterations. We denote $\mathbf{W}_p^{n+1}$ the deflection vector for p^{th} iteration at $(n+1)^{\text{th}}$ time step and approximate equation (13) as

$$\mathbf{W}_{p+1}^{n+1} = (\mathbb{M} + \frac{1}{2}\Delta t^2\mathbb{K})^{-1}\Big[\mathbf{G}^n - \rho\mathbb{R}\circ\Phi(\mathbf{W}_p^{n+1})\Big]. \quad (14)$$

$$\phi_p^{n+1} = \Phi(\mathbf{W}_p^{n+1}). \quad (15)$$

Equation (14) can be presented in abstract form

$$\mathbf{W}_{p+1}^{n+1} = \mathbb{H}(\mathbf{W}_p^{n+1}) \quad (16)$$

and the problem can be treated now as the fixed-point problem for the nonlinear operator $\mathbb{H}$.

By using a closed source FEM package, the operator $\mathbb{H}$ is realized as follows. We take values $\mathbf{W}^n$ and $\mathbf{W}^{n-1}$ and evaluate the *initial* data $\mathbf{W}(t_n)$ and $\mathbf{W}_t(t_n)$, where $t_n = n\Delta t$. Then we take an approximate distribution of the displacement potential ϕ_0^{n+1} and evaluate the pressure over the structure with the help of the Bernoulli equation

$p = -\rho[\phi_0^{n+1} - 2\phi^n + \phi^{n-1}]/\Delta t^2$. Next, we run a FEM code to compute the deflection $\mathbf{W}_1^{n+1}$ at the time instant t_{n+1}. After that we evaluate next approximation of the displacement potential ϕ_1^{n+1} by using (15) and update the loads acting on the structure. Continuing with iterations, we hope that the iterations converge to a deflection $\mathbf{W}^{n+1}$. After the convergence achieved, we go to the next time instant.

This is so-called decoupled algorithm. This algorithm has limited applications in the problems of hydroelastic impact, because in most interesting cases it does not converge. Here we described the algorithm in details because it is rather attractive (but wrong) and because in this algorithm the operator $\mathbb{H}$ is defined. This definition will be used in the following analysis.

3 Convergence of the algorithm

In order to study convergence of the algorithm, which is based on equation (16), we consider a one-degree of freedom elastic system. Physically, the system consists in a rigid body, which is submerged into an unbounded, ideal and incompressible liquid and restricted by a spring with stiffness k. Equation of the body motion has the form

$$m\ddot{x} + kx = -m_a\ddot{x}, \tag{17}$$

where the right hand side represents the hydrodynamic force acting on the body, m_a is the added mass of the body and $x(t)$ is the body displacement. Equation (17) has the same form as (11), where the operator Φ is linear now, $\Phi(\mathbf{W}) = m_a\mathbf{W}/\rho$, $\mathbf{W}(t)$ has the only component denoted as $x(t)$.
The operator $\mathbb{H}$ defined in (14) and (16) for the multi-degrees of freedom nonlinear model, is linear for the linear system (17):

$$\mathbb{H}(x) = \frac{1}{m_s}[G - m_a x], \quad m_s = m + \frac{1}{2}k\Delta t^2.$$

Convergence of the iteration algorithm (16) is determined by the ratio

$$K = \frac{|\mathbb{H}(x) - \mathbb{H}(x')|}{|x - x'|}. \tag{18}$$

If $K < 1$, the algorithm converges. In the case of equation (17), we find $K = m_a/m_s$, which is less than unity if the added mass of the body m_a is less than the body mass m.
Note that in (17) the inertia term $m\ddot{x}$ and the hydrodynamic force $-m_a\ddot{x}$ have the same form. The algorithm based on equation (16) assumes that the inertia term provides more important contribution to the equation than the hydrodynamic load term. This is true when $m_a < m$. The latter inequality is usually valid in aeroelasticity due to small air density. In the problem of hydroelastic impact, one may expect that the algorithm (16) converges at the very initial stage, when the wetted area of the structure is small and, correspondingly, the added mass is smaller than the mass of the structure per unit area. In the problems of strong interaction between elastic structures and liquid, the added mass of the structure is usually greater than the structural mass and the described algorithm diverges.

It is suggested to modify equation (16) as

$$\omega\mathbf{W}_{p+1}^{n+1} + (1-\omega)\mathbf{W}_p^{n+1} = \mathbb{H}(\mathbf{W}_p^{n+1}), \tag{19}$$

where $\omega \neq 0$ is a parameter. Equation (19) gives rise to a modified operator $\mathbb{H}'$

$$\mathbf{W}_{p+1}^{n+1} = \frac{\omega - 1}{\omega}\mathbf{W}_p^{n+1} + \frac{1}{\omega}\mathbb{H}(\mathbf{W}_p^{n+1}) = \mathbb{H}'(\mathbf{W}_p^{n+1}). \tag{20}$$

For the new operator $\mathbb{H}'$ we find

$$K'_\omega = \left|\frac{1}{\omega}\right| \cdot \left|\omega - 1 - \frac{m_a}{m_s}\right|. \tag{21}$$

Figure 2 plots K'_ω as a function of ω.

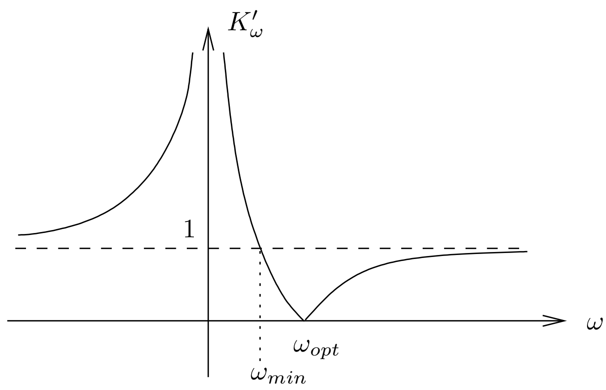

Figure 2: K'_ω as a function of ω

It is seen that there is a range of ω, where $K'_\omega < 1$. Moreover, there exists an *optimal* value ω_{opt}, for which $K'_\omega = 0$ and already the first iteration provides exact solution of the linear model problem. Numerical experience shows that an optimal ω, for which the convergence of iterations is fastest (*i.e.* K'_ω is minimal), exists also for multi-degrees of freedom nonlinear problems, but now ω_{opt} is dependent on the solution and is different for different time steps.

This one-degree of freedom study lets us realize convergence difficulties and lets us find a solution to resolve them. This simplified study does not provide an absolute proof of convergence of algorithm (20). In the present study we believe that by introducing the parameter ω and choosing its optimal value at each time step, we can arrive at convergence of algorithm (20) and reach good computational efficiency of the numerical simulations.

4 2D Numerical validation

A preliminary 2D numerical code was developed, in order to test the algorithm. The numerical scheme (20) for the coupled hydro-elastic impact problem was validated by using a semi-analytical solution obtained for symmetric wedge impact problem [4].

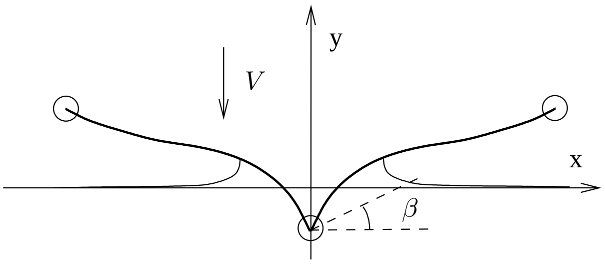

Figure 3: The symmetric wedge

One considers a two-dimensional symmetric simply supported wedge with deadrise angle $\beta = 5$ (see figure 3). The thickness of the wedge plating is 2 cm and the length is 1 m. The wedge is made of steel with density 7800 kg/m^3, Young modulus $210 \cdot 10^9\ Pa$ and Poisson ratio 0.3. Impact velocity is $4\ m/s$ and constant during the impact. The water density is 1000 kg/m^3.

The graph 4 presents the vertical deflections with respect to time for points situated at 50 cm of the wedge centre.

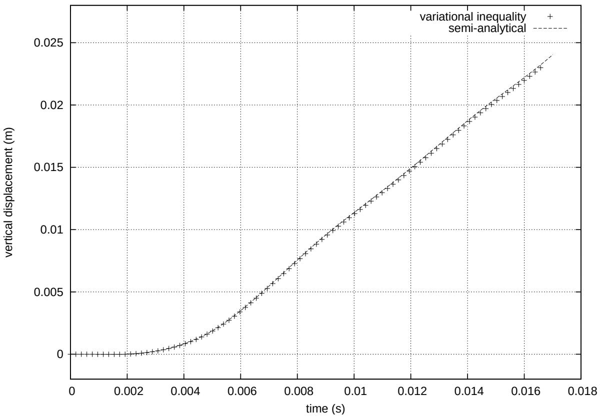

Figure 4: Vertical deflection with respect to time for the point situated at 50 cm of the centre

Graph 4 shows good agreement between the two methods. Absolute error and relative error have been computed; results are presented in graph 5.

One notice that the relative error is not small at the beginning of the impact. However, during the very early stage the vertical displacements are small and the absolute error is rather small too.
During the final stage of the impact, absolute error increases (the vertical deflection also increases), but the relative error is small ($\simeq 1\%$).

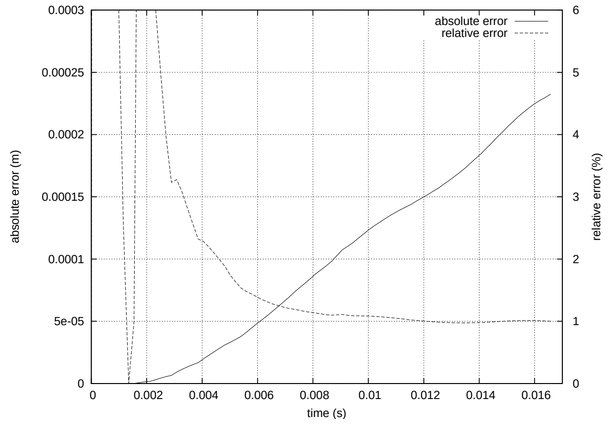

Figure 5: Absolute error (thick line) and relative error (dashed line) between semi-analytical results in [4] and the variational inequality results, for the point x=50 cm

Conclusions

A numerical method for solving the coupled Wagner problem of elastic structure impact has been proposed in this paper. The stability of the numerical scheme is studied, no artificial smoothing or filtering has been used to enforce convergence. The comparison between the present numerical results and semi-analytical solution shows good agreements.
This numerical method can be used for any finite element structure linearly elastic.
A future work consists in three dimensional implementation of this method and its coupling with a professional finite element package.

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Discusser - Y Kim:

Could you introduce your experience how to solve the temporal and spatial resolution difference? The structural model, ie FEM model, has different resolutions of time and mesh with those of the hydrodynamic model. According to my experience, to match these resolutions are not easy for some problems. How about your experience?

Reply:

The structure is meshed without taking into account the fluid. The fluid and structure meshes are not compatible at the interface. However, particular attention is paid to perform a very precise interpolation between the two meshes. The size of the structural mesh provides limitations on the time step. The fluid is assumed incompressible; a crucial point is that the expansion of the wet area must be well discretized and resolved. This introduces more conditions on the time step and also on the fluid mesh. Finally, we use a time step which satisfies all these conditions.

Discusser - K. Takagi:

If you use the wet mode instead of the dry mode, what would happen? This may solve the convergence problem.

Reply:

Wet part of entering body is unknown. Just after impact instant the wet area of the body is small, which makes use of the dry modes reasonable. When the body is already totally wet but still continue to interact with the fluid, then the use of wet modes is very reasonable.

It may be possible to compute partially wet modes. But to do this, we need to know exactly the wet area, which is part of the problem. So it is not possible to compute the exact partially wet modes, since we don't know the exact position of the fluid. An other point is that these partially wet modes should be recomputed at each time step.

However, it should be possible to approximate the partially wet modes (and to recompute them only several times during the impact), and use these approximations to compute the hydro-elastic problem. The convergence should then be improved.

Discusser - R.W. Yeung:

The boundary condition on the body neglects the tangential components on the surface of the body. The theory is thus applicable to very small dead rise angle. The variational form cannot be extended to include the more exact boundary condition, I presume. Is that so?

Reply:

The present variational formulation is based on Wagner formulation of impact problems. So it does not seem possible at present to extend this formulation to more complex boundary conditions.