Potential Flow below the Capillary Surface of a Viscous Fluid

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The potential flow in a viscous fluid due to a point impulsive force applying at the free surface is considered within the framework of linear Stokes equations. The combined effect of fluid viscosity and surface tension on the potential function below the water surface is studied. Dependent on the wavenumbers associated with the level of the effect due to surface tension, the oscillations can be grouped as gravity-dominant waves and capillary-dominant waves. It is shown that the wave form of gravity-dominant oscillations is largely modified by the surface tension while the wave amplitude of capillary-dominant oscillations is mostly reduced by the fluid viscosity.

1. Stokes equation

We consider the lower half-space filled with water limited on the top by the water-air interface. A Cartesian coordinate system is defined by placing the (x, y)-plane coincided with the undisturbed free surface and the z-axis oriented positively upward. In this gravity-dominant fluid domain, the reference length L, the acceleration of gravity g and the water density ρ are used to define the nondimensional coordinates $\mathbf{x} = (x, y, z)$, the time t, the fluid velocity $\mathbf{u} = (u, v, w)$, the velocity potential Φ, the dynamic pressure P and forces F with respect to $(L, \sqrt{g/L}, \sqrt{gL}, \sqrt{gL^3}, \rho g L, \rho g L^3)$, respectively. We study the flow due to a point impulsive force F applied vertically downward at the origin of coordinate system. By assuming the incompressibility, the fluid flow is governed by the continuity equation and the momentum equation :

$$\nabla \cdot \mathbf{u} = 0 \quad \text{and} \quad \mathbf{u}_t = -\nabla P + \epsilon \nabla^2 \mathbf{u} - \mathbf{e}_z F \delta(\mathbf{x}) \delta(t) \tag{1a}$$

where $\epsilon = \mu/(\rho\sqrt{gL^3})$ with μ the fluid viscosity. The term $-\mathbf{e}_z F\delta(x)$ represents the singular force located at the origin where $\mathbf{e}_z$ is the unit vector in the z direction, $\delta(\cdot)$ is the Dirac delta function.

On the free surface $z = \eta(x, y, t)$, the boundary conditions are linearized by assuming small wave amplitudes and written on the undisturbed free surface $z = 0$:

$$\eta_t = w \tag{2a}$$

as the kinematic condition stating no fluid particles cross the free surface and

$$\epsilon(u_z + w_x) = 0 = \epsilon(v_z + w_y) \tag{2b}$$

$$\eta - \sigma^2(\eta_{xx} + \eta_{yy}) + 2\epsilon w_z = P \tag{2c}$$

as the dynamic conditions representing the vanishing of shear stress in both x and y directions (2b) and the equation of normal stress (2c). In (2c), $\sigma = \sqrt{T/(\rho g L^2)}$ with T is the surface tension of water-air interface.

In addition, the initial values of the velocity, the hydrodynamic pressure and the free-surface elevation are taken as those of the quiescent fluid, i.e.

$$\mathbf{u} = P = \eta = 0 \quad \text{at} \quad t = 0 \tag{3}$$

The equations (1-3) construct an initial-boundary-value problem.

2. Solution of the initial-boundary-value problem

To solve the initial-boundary-value problem preceding defined (1-3), the unknowns $(\mathbf{u}, P)$ are decomposed as the sum of an unbounded singular Stokes flow $(\mathbf{u}^S, P^S)$ and the regular flow $(\mathbf{u}^R, P^R)$ which represents the free-surface effect. Furthermore, the continuous vector $\mathbf{u}^R$ is written as the sum of an irrotational and a solenoidal vectors :

$$\mathbf{u}^R = \nabla\Phi + \mathbf{u}^T \tag{4}$$

such that

$$\nabla^2\Phi = 0 = \nabla \cdot \mathbf{u}^T \quad \text{and} \quad \mathbf{u}_t^T = \epsilon \nabla^2 \mathbf{u}^T \tag{5}$$

where the scalar function $\Phi(\mathbf{x},t)$ represents the irrotational flow while $\mathbf{u}^T$ the rotational flow. The dynamic pressure P^R is defined by :

$$P^R = -\Phi_t + f(t) \tag{6}$$

where the function $f(t)$ is introduced to satisfy the initial condition $P = P^R + P^S = 0$ at $t = 0$ since $\Phi^0 = \Phi(t=0)$ may not be necessary zero. Indeed, we have used $f(t) = -\Phi^0\delta(t)$ directly in the following for the sake of simplicity.

The boundary conditions (2) can now be expressed in terms of $(\mathbf{u}^S, P^S, \Phi, \mathbf{u}^T)$ on the undisturbed free surface $(z=0)$:

$$\eta_t - (\Phi_z + w^T) = w^S \tag{7a}$$
$$2\Phi_{zx} + u_z^T + w_x^T = -(u_z^S + w_x^S) \tag{7b}$$
$$2\Phi_{zy} + v_z^T + w_y^T = -(v_z^S + w_y^S) \tag{7c}$$
$$\Phi_t + \eta - \sigma^2(\eta_{xx} + \eta_{yy}) + 2\epsilon(\Phi_{zz} + w_z^T) = P^S - 2\epsilon w_z^S - \Phi_0^0\delta(t) \tag{7d}$$

in which $\Phi_0^0 = \Phi(x,y,z=0,t=0)$. The unbounded singular flow on the right hand side of (7) is well known in the work of Lu & Chwang (2004) and satisfies :

$$P^S = F/(4\pi)\,\partial_z(1/|\mathbf{x}|)\delta(t) \tag{8a}$$

$$\mathbf{u}_t^S - \epsilon\nabla^2\mathbf{u}^S = -F/(4\pi)\big[(\partial_{zx}, \partial_{zy}, -\partial_{xx} - \partial_{yy})(1/|\mathbf{x}|)\big]\delta(t) \tag{8b}$$

which yield the solutions in integral forms for $(\mathbf{u}^S, P^S)$ by taking the Laplace transform with respect to t and the Fourier integral with respect to (x,y). In the same way, we introduce the joint integral transform for $(\eta, \Phi, \mathbf{u}^T)$ as :

$$[\tilde{\eta}, \tilde{\Phi}, \tilde{\mathbf{u}}^T] = \int_0^\infty dt \int_{-\infty}^\infty dy \int_{-\infty}^\infty dx\, [\eta, \Phi_0 e^{kz}, \mathbf{u}_0^T e^{k_\epsilon z}] e^{-i(\alpha x + \beta y) - st} \tag{9}$$

in which, we have used the notations :

$$\Phi_0 = \Phi(x,y,z=0,t)\ ; \quad \mathbf{u}_0^T = \mathbf{u}^T(x,y,z=0,t)\ ; \quad k = \sqrt{\alpha^2+\beta^2} \quad \text{and} \quad k_\epsilon = \sqrt{s/\epsilon + k^2}$$

Taking the joint integral transform (9) over the left hand side of (7) as well as $\nabla\cdot\mathbf{u}^T = 0$, and introducing the integral form of $(P^S, \mathbf{u}^S)$ on the right hand side of (7), a system of linear equations is obtained for the five unknowns :

$$\begin{pmatrix} s & -k & 0 & 0 & -1 \\ 0 & 2ik\alpha & k_\epsilon & 0 & i\alpha \\ 0 & 2ik\beta & 0 & k_\epsilon & i\beta \\ 1+\sigma^2k^2 & s+2\epsilon k^2 & 0 & 0 & 2\epsilon k_\epsilon \\ 0 & 0 & i\alpha & i\beta & k_\epsilon \end{pmatrix} \begin{pmatrix} \tilde{\eta} \\ \tilde{\Phi}_0 \\ \tilde{u}_0^T \\ \tilde{v}_0^T \\ \tilde{w}_0^T \end{pmatrix} = -\frac{F}{2s} \begin{pmatrix} k(1-k/k_\epsilon) \\ i\alpha k_\epsilon(1-k/k_\epsilon)^2 \\ i\beta k_\epsilon(1-k/k_\epsilon)^2 \\ s \\ 0 \end{pmatrix} \tag{10}$$

which gives :

$$\tilde{\Phi}_0 = -\frac{F}{2}\left(\frac{N}{D} - \frac{1}{s}\right) \quad \text{with} \quad N = 2(s+2\epsilon k^2) \quad \text{and} \quad D = \omega^2 + (s+2\epsilon k^2)^2 - 4\epsilon^2k^3k_\epsilon \tag{11}$$

In (11), D is often called the dispersion function and $\omega^2 = k + \sigma^2k^3$. Similar results for $(\tilde{\eta}_0, \tilde{u}_0^T, \tilde{v}_0^T, \tilde{w}_0^T)$ with the same dispersion function D in denominator can be obtained. The wave elevation η has been considered in a number of studies as in Miles (1957) and in Lu & Chwang (2003). Since the amplitude functions of $(\tilde{u}_0^T, \tilde{v}_0^T, \tilde{w}_0^T)$ are of order $(\epsilon^{1/2}, \epsilon^{1/2}, \epsilon)$, respectively, we are interested here to the potential function Φ which is obtained by taking the inverse integral transform :

$$\Phi = -\frac{F}{16\pi^3 i}\int_{c-i\infty}^{c+i\infty} ds \int_{-\infty}^\infty d\beta \int_{-\infty}^\infty d\alpha\, (N/D - 1/s) e^{kz + i(\alpha x + \beta y) + st} \tag{12}$$

where N and D are given in (11) and c is the Laplace convergence abscissa in the Bromwich integral for the inversion of Laplace transform.

3. Evaluation of the potential function

To evaluate the potential function $\Phi(\mathbf{x},t)$, we examine the equation $D = 0$ which gives two poles of the Laplace integral :

$$s^\pm = \pm i\omega - 2\epsilon k^2 + O(\epsilon^{3/2}) \tag{13}$$

By taking a contour integration in the complex s plane using the Cauchy residue theorem, we obtain :

$$\Phi(x,y,z,t) = -\frac{F}{8\pi^2}\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \left[2e^{-2\epsilon k^2 t}\cos(\omega t) - 1\right] e^{kz+i(\alpha x+\beta y)} + O(\epsilon) \tag{14}$$

which can be further written by introducing the Bessel function $\mathrm{J}_0(\cdot)$ of the first kind :

$$\Phi(h,z,t) = -F/(4\pi)\int_0^{\infty} dk\, k\left[2e^{-2\epsilon k^2 t}\cos(\omega t) - 1\right] e^{kz}\mathrm{J}_0(kh) + O(\epsilon) \tag{15}$$

in which $h = \sqrt{x^2+y^2}$. The derivative of Φ with respect to t is directly derived from (15) and written as

$$\Phi_t(h,z,t) = F/(2\pi)\int_0^{\infty} dk\; e^{-2\epsilon k^2 t} k\,\omega \sin(\omega\tau)\, e^{kz}\mathrm{J}_0(kh) + O(\epsilon) \tag{16}$$

Following the work by Chen & Duan (2003) in which a potential function similar to (16) but in an inviscid fluid was analyzed, there are two saddle points k_g and k_T associated with the phase function $\psi = \omega - ka$ of the oscillatory part of the integrand in (16) :

$$k_g = 1/(4a^2) + O(\sigma/a^2) \quad \text{and} \quad k_T = 4a^2/(9\sigma^2) + O(\sigma/a^2) \quad \text{for} \quad a \gg \sqrt{\sigma} \tag{17}$$

where $a = h/t$ is the wave velocity. When a is of the same order as $\sqrt{\sigma}$, the wavenumbers k_g and k_T become close and in particular, $k_g = k_T = k_0 \approx 0.393/\sigma$ for $a = a_0 \approx 1.086\sqrt{\sigma}$. When $a < a_0$, the wavenumbers k_g and k_T are complex. This analysis shows that there are two waves propagating at the same speed a : one is gravity-dominant wave with a lower wavenumber k_g (17) and another capillary-dominant wave with a much larger wavenumber k_T (17). There is also a minimum speed a_0 below which two waves become evanescent.

Unlike the inviscid potential function in Chen & Duan (2003), the factor $e^{-2\epsilon k^2 t}$ is present in the integrand of (16). This exponentially-decreasing function reduces the amplitude of Φ_t at large time and for waves of large wavenumbers. The capillary-dominant waves are then heavy damped by the viscous effect while gravity-dominant waves are much less affected at small values of time.

To confirm above analysis, we have performed the numerical computation of (16) in the complex k-plane by using the steepest descent algorithm. The figures on the next page illustrate Φ_t for $F/(-2z\pi) = 1$ and $z = -1/1000$ at a fixed $t = 10$ and h varying from 0 to 8/5.

4. Discussions and conclusions

The potential function Φ_t defined by (16) with the fluid viscosity but without taking account of the effect of surface tension is shown on Figure 1. Large waves with small wavelength are present at a region of small distance from the singularity. Due to the viscous effect and the immersion $z = -1/1000$, the wave amplitude of larger wavenumbers close to the singularity is reduced to zero. This is consistent with that the transient pure-gravity waves on the free surface at a given instant oscillate with increasing amplitude and decreasing wavelength when we approach to the impulsive force point, as stated in Lamb (1932). Furthermore, the amplitude of pure-gravity waves increases linearly with time in a rate of order $O(t/h^2)$ and wavenumber increases in an order of $O[t/(4h^2)]$. This peculiar property of pure-gravity potential hinders the numerical development to solve the boundary-value problem associated with a floating body in which the space integral over body's surface as well as the time-convolution integral are difficult to be accurate.

Taking into account of the effect of surface tension, the wave form changes, in particular, there is not wave at all at small distance when $h/t < a_0$. On the other side, at larger distance from the impulsive point, the capillary-dominant waves have very large amplitudes with wavenumbers proportional to $4h^2/(9t^2\sigma^2)$ as shown on Figure 2. These large and short waves of capillarity are fortunately heavy damped by the viscous effect as shown on Figure 3.

In practice, the most interesting are the gravity-dominant waves. Their properties of the potential function with the combine effect of surface tension and fluid viscosity are welcome and believed to be much useful in the numerical solution of wave-body problems.

References

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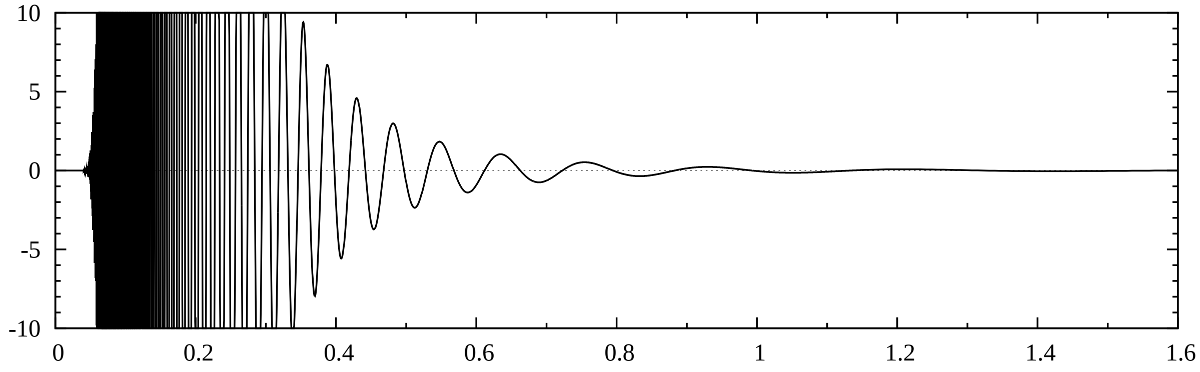

Figure 1: Potential function $\Phi_t(h, z, t)$ at $z = -1/1000$ and $t = 10$ against h varying from 0 to 8/5, obtained by taking only account of the fluid viscosity ($\epsilon = 3.193\text{e-}7$) but without the surface tension ($\sigma = 0$).

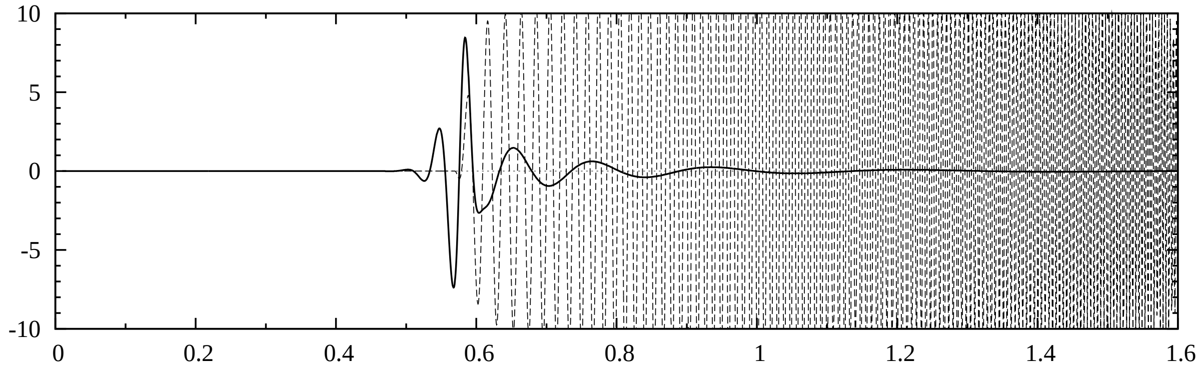

Figure 2: Gravity-dominant waves (solid line) and capillary-dominant waves (dashed line) of potential function $\Phi_t(h, z, t)$ at $z = -1/1000$ and $t = 10$ against h varying from 0 to 8/5, obtained by taking account of surface tension (σ=2.713e-3) but without the fluid viscosity ($\epsilon = 0$).

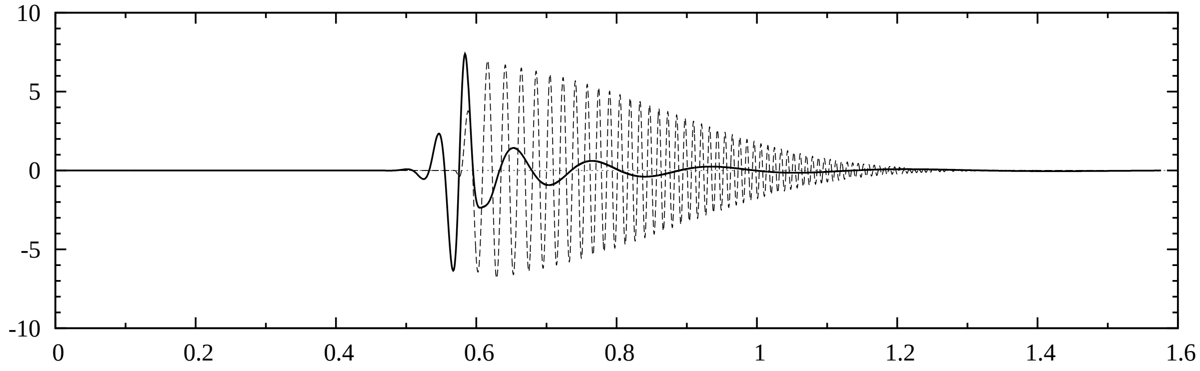

Figure 3: Gravity-dominant waves (solid line) and capillary-dominant waves (dashed line) of potential function $\Phi_t(h, z, t)$ at $z = -1/1000$ and $t = 10$ against h varying from 0 to 8/5, obtained by taking account of both the surface tension (σ=2.713e-3) and the fluid viscosity (ϵ=3.193e-7).