The proper definition of the added mass for the water entry problem

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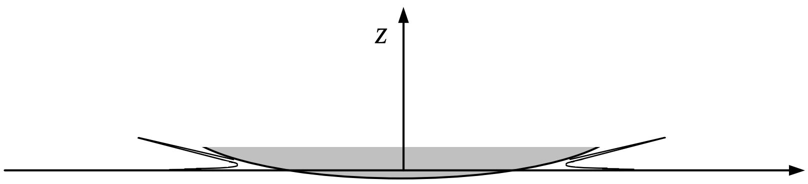
Introduction

The water entry problem is considered from the point of view of analytic mechanics. It is usual practice to treat potential hydrodynamic problems involving motion of solid bodies within the frame of system dynamics. This is done whenever a finite number of generalized coordinates can be used as a proper representation for the motion of the whole fluid. We refer to this kind of approach as 'hydro-mechanical'. This is made possible through the use of the well-known concept of added mass. In the vertical water entry problem the generalized coordinate is the penetration depth of the impacting body.

From another point of view, a usual way to calculate the impact force is the integration of the pressure-field over the wetted surface of the impacting body. The jets are then excluded from the portion of the fluid that wets the body.



For simplicity we take the case of vertical entry only. Under the hydro-mechanical approach, one may consider only the bulk of the fluid to express the impact force as

$$F = -\frac{\mathrm{d}}{\mathrm{d}t}\left(M_{bulk}U\right), \tag{1}$$

where M_{bulk} is the added mass associated to the kinetic energy of the bulk of the fluid, jets being excluded. However, if the added mass is interpreted as a measure of the kinetic energy of the whole fluid, another correct and alternative form for the impact force is given by

$$F = -\frac{\mathrm{d}}{\mathrm{d}t}\left(M_{wfd}U\right) + \frac{1}{2}\frac{\mathrm{d}M_{wfd}}{\mathrm{d}t}U, \tag{2}$$

where M_{wfd} refers to the *whole fluid domain*. Should *equivocal arguments* be used one would obtain the *misleading* expression,

$$F = -\frac{\mathrm{d}}{\mathrm{d}t}\left(M_{bulk}U\right) + \frac{1}{2}\frac{\mathrm{d}M_{bulk}}{\mathrm{d}t}U \,. \tag{3}$$

The reason for the discrepancy between expressions (1) and (3) has already been addressed and actually emerges from a consistent energy balance; see, e.g., the discussion in Molin *et al*. (1996) or in Pesce (2003, 2006). In fact, via an asymptotic analysis near the contact line, it has been shown, for some particular cases, that a considerable amount[1] of kinetic energy is drained from the bulk of the fluid through the jets; Molin *et al*. (1996), Scolan and Korobkin (2003), Casetta and Pesce (2005). In other words, the jets must be consistently considered if energy arguments are applied in the impact force calculation.

In the present work the role of the jets in the impact force calculation is discussed. We show how one can obtain the correct impact force expression in either way, by considering the jets, or by not considering them. In this sense, proper definitions of the added mass for the problem are presented and then used in the corresponding proper forms of the Lagrange equation.

Analysis considering the bulk of the fluid domain

A usual and practical approach in marine hydrodynamics is to treat the water entry problem by taking the domain of analysis as the bulk of the fluid only. The whole fluid domain is therefore divided into two parts: the bulk and the jets. As the body penetrates the bulk, jets are expelled from the neighborhood of the contact line due to a very rapid expansion of the body wetted surface. The kinetic energy of the bulk of the fluid, $T_{bulk} = \frac{1}{2} M_{bulk}U^2$, varies explicitly with the penetration depth of the body, and so does the added mass associated to it. In this case, where an out-flux of kinetic energy does exist from the domain under analysis (the bulk of fluid), there is an ‘effective loss of added mass’ through the jets and one must use the extended Lagrange equation valid for systems with mass explicitly dependent on position - see Pesce (2003) - that reads

$$F + \frac{1}{2}\frac{\mathrm{d}M_{bulk}}{\mathrm{d}z}U^2 = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_{bulk}}{\partial U} + \frac{\partial T_{bulk}}{\partial z} \,. \tag{4}$$

The left-hand side[2] of equation (4) must be interpreted as the total force that acts on the bulk of the fluid. The first term is the impact force we want to calculate. The second term, $\frac{1}{2}\left(\mathrm{d}M_{bulk}/\mathrm{d}z\right)U^2$, is the reactive force acting on the bulk due to the out-flux of kinetic energy. In fact, Casetta and Pesce (2005) show, for the case of a generic and arbitrary shape of the contact line, that the rate of kinetic energy drained by the jets is always equal to $\frac{1}{2}\left(\mathrm{d}M_{bulk}/\mathrm{d}z\right)U^3$, being the reactive force term $\frac{1}{2}\left(\mathrm{d}M_{bulk}/\mathrm{d}z\right)U^2$ also generally valid. Therefore, as $\partial T_{bulk}/\partial z = \frac{1}{2}\left(\mathrm{d}M_{bulk}/\mathrm{d}z\right)U^2$, one easily obtain the impact force in the form of equation (1), i.e.,

$$F = -\frac{\mathrm{d}}{\mathrm{d}t}\left(M_{bulk}U\right). \tag{1a}$$

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[1] Actually this amount is exactly half the whole kinetic energy of the fluid if the impact velocity is enforced to be constant; see Molin *et al.* (1996), for the case of cylinders; Casetta and Pesce (2005), for the general 3D case.

[2] Such an extended form of the Lagrange equation should also contain a reactive force term due to a real (physical) mass out-flux. In the water entry problem, the real mass out-flux through the jets is of second order, as shown by Cointe and Armand (1978) and Molin *et al*. (1996), for the particular and important case of a circular cylinder.

Analysis considering the whole fluid domain

The water entry problem can also be treated by taking the whole fluid domain. The added mass M_{wfd}, is now a measure of the kinetic energy of the whole fluid domain, i.e., $T_{wfd} = \frac{1}{2} M_{wfd} U^2$. In this case, the jets are included in the analysis domain and, obviously, there is no out-flux of kinetic energy – neither an 'out-flux of added mass'. In other words, there is no loss of energy from the system and this is the key point. The extended Lagrange equation, for systems with mass explicitly dependent on position is no longer applicable. One must then apply the usual form[3] of the Lagrange equation, as in Lamb (1932), art. 136 and 137, i.e.,

$$F = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_{wfd}}{\partial U} + \frac{\partial T_{wfd}}{\partial z}. \tag{5}$$

The impact force expression is then obtained from equation (5) in the form of equation (2),

$$F = -\frac{\mathrm{d}}{\mathrm{d}t}\left(M_{wfd} U\right) + \frac{1}{2}\frac{\mathrm{d}M_{wfd}}{\mathrm{d}t} U \,. \tag{2a}$$

The equivalence between impact force expressions

By equating the alternative equations (1) and (2), we promptly obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M_{wfd} U\right) - \frac{1}{2}\frac{\mathrm{d}M_{wfd}}{\mathrm{d}t} U = \frac{\mathrm{d}}{\mathrm{d}t}\left(M_{bulk} U\right). \tag{6}$$

Should a misleading assumption be taken, by enforcing $M_{bulk} = M_{wfd}$, an obviously *erroneous result* would be found, i.e.,

$$\frac{1}{2}\frac{\mathrm{d}M_{bulk}}{\mathrm{d}t} U = 0 \,, \tag{7}$$

such that $\frac{1}{2}\left(\mathrm{d}M_{bulk} / \mathrm{d}t\right) U^2$ would be also null. This would imply the amount of the kinetic energy in the jets to be null, Casetta and Pesce (2005), an obviously *false assertive*.

Energy balance

From equations (1) and (2) one can easily verify that

$$FU + \frac{\mathrm{d}T_{bulk}}{\mathrm{d}t} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(M_{bulk}\right) U^2 = 0\,, \tag{8}$$

and that

$$FU + \frac{\mathrm{d}T_{wfd}}{\mathrm{d}t} = 0\,. \tag{9}$$

[3] Recall that the usual form of Lagrange equation is invariant with respect to systems with mass varying as a function of time - as is the case if the whole fluid is taken as the domain; see, e.g., Pesce (2003), for a detailed discussion on this subject.

Both equivalent equations (8) and (9) represent the correct balance of energy between the fluid and the impacting body. The third term in equation (8) is the time rate of the kinetic energy that fills in the jet. It is also interesting to mention that the general equation (8) can be alternatively obtained via the classical velocity potential approach, by considering the water entry problem as a nonlinear boundary-value problem; see Casetta and Pesce (2005).

Conclusion

It was shown how to treat the water entry problem in two equivalent and consistent ways. If one considers only the bulk of the fluid as the domain of analysis, such that there is an out-flux of kinetic energy to the jets, and if an analytic mechanics point of view is followed, the extended form of the Lagrange equation, for systems with mass explicitly dependent on position, must be applied to calculate the impact force. Alternatively, if the whole fluid domain is taken, so that no longer exists an out-flux of kinetic energy, the usual form of the Lagrange equation must be applied. The consistency between both approaches lies on a proper domain definition of the added mass.

Acknowledgments

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References

Casetta, L., Pesce, C.P., 2005, "A Noticeable Question of the Water Entry Problem: the Split of Kinetic Energy during the Initial Stage", COBEM2005, 18th International Congress of Mechanical Engineering, Brazilian Society of Engineering and Mechanical Sciences, Ouro Preto, Brazil.

Cointe, R. and Armand, 1987, "Hydrodynamic Impact Analysis of a Cylinder", Journal of Offshore Mechanics and Arctic Engineering, Vol. 109, pp. 237-243.

Cointe, R., Fontaine, E., Molin, B. and Scolan, Y.M., 2004, "On Energy Arguments Applied to the Hydrodynamic Impact Force", Journal of Engineering Mathematics, Vol.48, pp. 305-319.

Lamb, H., 1932, "Hydrodynamics", Dover Publications, 6th ed., 738 pp.

Molin, B., Cointe, R. and Fontaine, E., 1996, "On Energy Arguments Applied to the Slamming Force", 11st.International Workshop on Water Waves and Floating Bodies, Hamburg.

Pesce, C.P., 2003, "The Application of Lagrange Equations to Mechanical Systems with Mass Explicitly Dependent on Position", Journal of Applied Mechanics, Vol. 70, pp. 751-756.

Pesce, C.P., 2006, "A Note on the Classical Free Surface Hydrodynamic Impact Problem", World Scientific, Hydrodynamics Symposium in honor to T.-Y. Wu, to appear.

Scolan, Y.M. and Korobkin, A.A., 2003, "Energy Distribution from Vertical Impact of a Three-Dimensional Solid Body onto the Flat Free Surface of an Ideal Liquid", Journal of Fluids and Structures, Vol.17, pp. 275-286.

Wu, G. X., 1998, "Hydrodynamic Force on a Rigid Body during Impact with Liquid", Journal of Fluids and Structures, Vol. 12, pp. 549-559.

Casetta, L., Pesce, C.P.
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Discusser - R.C.T. Rainey:

I disagree with your equation (1). If the fluid has infinite depth, the fluid momentum is not defined, see beginning of chapter 6 in Lamb (1932). If the fluid has finite depth, there is a reaction force on the seabed, which does not tend to zero as the depth increases.

Reply:

First, we agree that in the case of finite depth there would be a reaction force on the seabed. Moreover the added mass would be also a function of the proximity to the seabed. Even in that finite depth case, however, the fluid domain might be unbounded. We note that we are treating the infinite depth case only. The momentum (and the kinetic energy) of the liquid may be defined whenever the velocity field is integrable (square integrable, i.e, integrable in the energy norm) over the whole liquid domain. This is the case in the present paper.