An approximation to two-dimensional wave scattering by topography using orthogonal curvilinear coordinates

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1 Introduction

Two-dimensional linear wave scattering by topography has received extensive attention in the literature. Such situations are commonly addressed by approximating the vertical structure of the fluid motion and then "averaging" over the depth to remove the vertical coordinate. In its simplest form this approach yields shallow-water theory, but also includes the so-called mild-slope (Berkhoff, 1976) and modified mild-slope equations (Chamberlain & Porter, 1995).

Here we follow the spirit of the averaging methods, but first rewrite the governing equations in terms of new orthogonal curvilinear coordinates, chosen so that the free surface and bed profile coincide with coordinate lines of the new 'vertical' coordinate, ζ. The resulting system is then dealt with by the averaging procedure described above, but where it is now the ζ dependence which is approximated and then averaged, and the governing ordinary differential equation is in the new 'horizontal' coordinate ξ. One advantage of this transformation is that the no normal flow condition at the bed transforms to a simpler Neumann condition in the new coordinates, and the ζ dependence can be chosen to satisfy this condition exactly. The flow field near the bed can thus be accurately reproduced with only a one-term approximation, since at the bed the lines $\xi =$ constant and the bed profile are perpendicular. This procedure thus avoids the need to include the so-called "bed mode" of Athanassoulis & Belibassakis (1999) and Chamberlain & Porter (2006).

2 Preliminaries

Incompressible and homogeneous fluid undergoes irrotational motion above a bed of varying depth $z = -h(x)$ for $h > 0$, and where x and z are horizontal and vertical Cartesian coordinates respectively, with $z = 0$ coinciding with the undisturbed fluid free surface. Harmonic time-dependence is filtered from proceedings by writing the velocity potential Φ

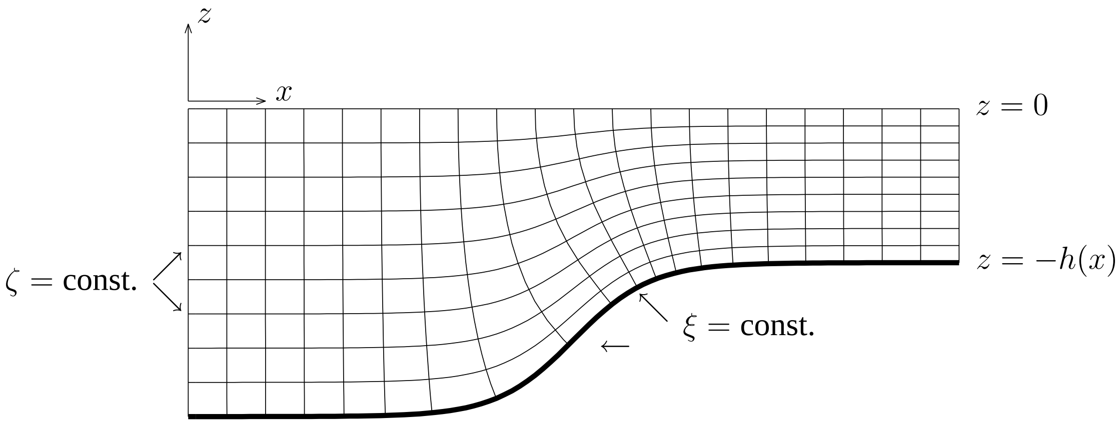

Figure 1: Transformed coordinates (ξ, ζ) for a typical bed profile $z = -h(x)$.

as

$$\Phi(x, z, t) = \mathrm{Re}\{\phi(x, z)e^{-i\omega t}\}$$

where ω is an assigned wave frequency. The usual equations of linearised wave theory then hold, namely

$$\nabla^2\phi = 0 \qquad (-h < z < 0), \tag{2.1}$$

$$\phi_z - K\phi = 0 \qquad (z = 0), \tag{2.2}$$

$$\phi_z + h'\phi_x = 0 \qquad (z = -h), \tag{2.3}$$

where $K = \omega^2/g$ and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$. In addition, appropriate radiation conditions are required to ensure that the velocity potential ϕ is unique, and these are specified below.

3 Variable transformation

We introduce the new 'vertical' coordinate

$$\zeta = \frac{zH}{h}, \tag{3.1}$$

where H is a constant scaling factor to be chosen. The free surface $z = 0$ and bed $z = -h$ thus correspond to $\zeta = 0$ and $\zeta = -H$, respectively (see figure 1). At every point (x, z) the new 'horizontal' coordinate, ξ, is perpendicular to ζ, so $\nabla\xi \cdot \nabla\zeta = 0$, and ξ is thus determined from

$$zh'(x)\xi_x - h(x)\xi_z = 0.$$

In particular, ξ is constant on characteristics satisfying

$$\frac{\mathrm{d}x}{\mathrm{d}z} = -\frac{zh'(x)}{h(x)}.$$

It remains to choose a suitable boundary condition for ξ. One possibility is to choose ξ such that

$$\xi = \int^x \frac{H}{h(s)}\,\mathrm{d}s \qquad (z = 0). \tag{3.2}$$

This choice has the advantage that Laplace's equation is retained in regions of constant fluid depth (see equation (3.3), below), but the numerical approximation of the integral in (3.2) adds an extra level of computational difficulty. For simplicity we thus choose x and ξ to coincide on the free surface $z = \zeta = 0$. Then, given (x, z), ξ is determined by

$$\xi = \hat{x}(0), \quad \text{where} \;\; \frac{\mathrm{d}\hat{x}}{\mathrm{d}\hat{z}} = -\frac{\hat{z}h'(\hat{x})}{h(\hat{x})}, \;\; \hat{x}(z) = x.$$

In terms of the new coordinates, Laplace's equation (2.1) transforms to

$$\frac{\partial}{\partial \xi}\left(\alpha \frac{\partial \phi}{\partial \xi}\right) + \frac{\partial}{\partial \zeta}\left(\frac{1}{\alpha}\frac{\partial \phi}{\partial \zeta}\right) = 0, \tag{3.3}$$

where $\alpha = h\xi_x/H$, whilst the boundary conditions (2.2) and (2.3) become

$$\phi_\zeta - \frac{Kh}{H}\phi = 0 \qquad (\zeta = 0) \tag{3.4}$$

and

$$\phi_\zeta = 0 \qquad (\zeta = -H), \tag{3.5}$$

respectively. In regions of constant depth, $\xi = x$, so $\alpha = h/H$ is constant and (3.3) in conjunction with (3.4) and (3.5) becomes separable. Propagating solutions are of the form

$$\phi = \left\{Ae^{i\kappa H\xi/h} + Be^{-i\kappa H\xi/h}\right\}\cosh[\kappa(\zeta + H)] \equiv \left\{Ae^{ikx} + Be^{-ikx}\right\}\cosh[k(z + h)], \tag{3.6}$$

where $\kappa = kh/H$ and k is the real and positive solution of the usual dispersion relation

$$K = k\tanh(kh). \tag{3.7}$$

This solution for $\phi = \phi(\xi, \zeta)$ forms the basis of our approximation in fluid regions above non-uniform bed sections.

We suppose that the bed profile function $h(x)$ satisfies

$$h(x) = \begin{cases} h_0 & x \le 0 \\ h_l & x \ge l \end{cases}$$

for some $l > 0$, where h_0 and h_l are positive constants, and that $h'(0) = h'(l) = 0$. (The scaling factor H in (3.1) is then chosen as $H = h_0$.) In the regions of constant fluid depth we write $\phi(x, z) = \cosh[k(z + h)]\mathrm{sech}(kh)\phi_0(x)$, where

$$\phi_0(x) = \begin{cases} A_- e^{ik_0 x} + B_- e^{-ik_0 x} & x \le 0, \\ A_+ e^{-ik_l(x-l)} + B_+ e^{ik_l(x-l)} & x \ge l. \end{cases} \tag{3.8}$$

Here, k_0 and k_l are the solutions of the dispersion relation (3.7) in $x < 0$ and $x > l$, respectively, A_- and A_+ are known, and B_- and B_+ are to be determined.

In the region of non-uniform bed, we approximate ϕ by $\phi \approx \psi$, and seek a Galerkin approximation to (3.3), by requiring that $\nabla^2\psi$ is orthogonal to some given function w. Thus

$$\int_{-\xi_0}^{\xi_0}\int_{-H}^{0}\left\{\frac{\partial}{\partial \xi}\left(\alpha\frac{\partial \psi}{\partial \xi}\right) + \frac{\partial}{\partial \zeta}\left(\frac{1}{\alpha}\frac{\partial \psi}{\partial \zeta}\right)\right\} w \,\mathrm{d}\zeta\,\mathrm{d}\xi = 0, \tag{3.9}$$

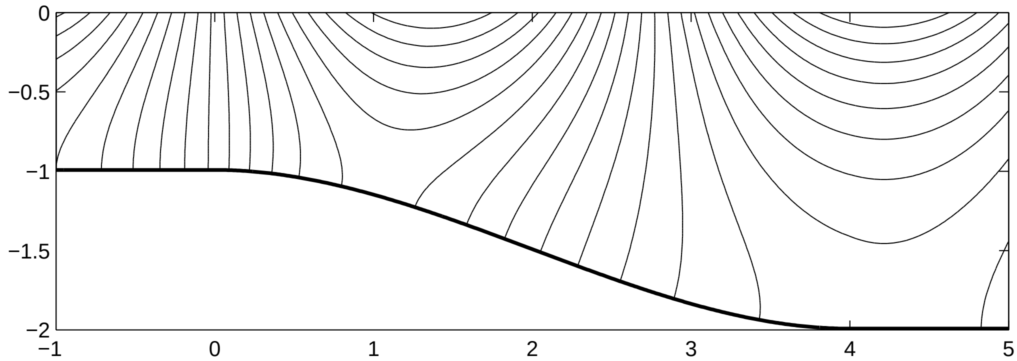

Figure 2: Contours of $\mathrm{Im}(\phi)$ in the case of a plane wave from $x = -\infty$ incident on the bed profile $h(x) = h_l - (h_l - h_0)(1 + 2(x/l)^3 - 3(x/l)^2)$, with $l = 4$, $h_0 = 1$ and $h_l = 2$.

for some $\xi_0 > 0$. Following the structure of the propagating mode (3.6) for regions of uniform depth, we choose $\psi = \phi_0(\xi)w(\xi,\zeta)$, where $w = \cosh[\kappa(\zeta + H)]$, and (3.9) may then be manipulated to show that $\phi_0(\xi)$ satisfies

$$(u(\xi)\phi_0'(\xi))' + v(\xi)\phi_0(\xi) = 0 \tag{3.10}$$

where

$$u(\xi) = \int_{-H}^{0} \alpha w^2 \, \mathrm{d}\zeta, \qquad v(\xi) = \int_{-H}^{0} \left\{ w(\alpha w_\xi)_\xi - \frac{1}{\alpha} w_\zeta^2 \right\} \mathrm{d}\zeta + \frac{1}{\alpha} w w_\zeta \bigg|_{\zeta=0}.$$

Equation (3.10) is solved numerically in the region $\xi \in (0, l)$, and continuity of ϕ_0 and ϕ_0' is imposed across $x = 0$ and $x = l$ to match the uniform depth solutions (3.8).

4 Results

Figure 2 shows contours of ϕ for a particular bed profile and incident wave. Note that the equipotentials intersect the bed profile at right-angles, in contrast to other single mode depth-averaging methods.

A selection of further results will be presented in the talk.

References

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