WAVEGUIDE PROPERTIES OF THE ELONGATED RECTANGULAR STRUCTURES

I. V. Sturova, Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russia

1. INTRODUCTION

It is well known that some infinite structures, like the submerged horizontal cylinder and the underwater mountain ridge, can support trapped modes. Evans & Kuznetsov (1997) gave a detailed review of the theoretical developments achieved in the recent decades in the studies on the existence of trapped waves within the context of linearized theory of waves in finite depth water as well as deep water. For sufficiently long waves it is possible to adopt the linearized shallow-water equations. This simplification enables a number of explicit solutions for trapped modes to be constructed for particular bottom geometries (see *e.g.* Le Blond & Mysak, 1978).

The phenomenon of wave trapping is referred to waves which can travel unchanged along the infinite 3D structure, and decay exponentially in the transverse direction. Our interest is in studying of the manifestation of trapped modes revealed for an infinitely long structure in the case of the corresponding finite-length structure. Two problems are considered:
1) a floating elastic plate;
2) a bottom topography with a piecewise-constant fluid depth.

The waves are generated by an external periodic pressure acting either on an elastic plate (problem 1) or on a free surface over a bottom topography (problem 2). The fluid is assumed to be incompressible, inviscid and its motion irrotational. The frequency of external loading is such that the length of generated waves is significantly greater than the depth of the fluid and the linear shallow-water theory can be used. The boundary-integral-equation method is applied and the action of the periodic surface pressure can be considered for an elastic plate (a bottom topography) in problem 1(2) of arbitrary planform. However, the main attention is paid to the rectangular planform.

2. AN ELASTIC FLOATING PLATE

An elastic plate of rectangular plan geometry, with length L and breadth B is considered, see Fig. 1. The plate is freely floating on a fluid layer of constant density ρ and depth h. Two regions which correspond to the fluid-plate region Ω_1 and the fluid region Ω_2 are distinguished. These regions are separated by the juncture boundary S.

The external periodic pressure of the form

$$p(x, y, t) = P(x, y)\exp(-i\omega t)$$

acts on the plate surface, where x and y are the horizontal coordinates with the origin at the center of the plate, t is time. The dark circle in Fig. 1 indicates the region of external pressure application. Within the scope of the linear theory, the motions of the plate and fluid can be assumed to be periodic in time with the same frequency ω. Hereafter, we will represent any time-harmonic function, say $f(x, y, t)$ as the real part of $F(x, y)\exp(-i\omega t)$ by introducing a complex function $F(x, y)$ that depends on the spatial variables only. The velocity potentials describing the fluid motion in the regions Ω_1 and Ω_2 are denoted by $\Phi_1(x, y)$ and $\Phi_2(x, y)$, respectively. The spatial part of the elevation of the water surface or the deflection of the structure are given by a complex function $W(x, y)$.

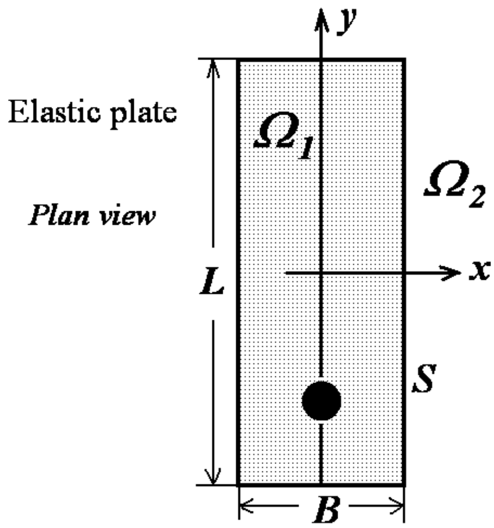

Figure 1: Definition sketch

The deflection of the plate is assumed to be governed by the equation

$$D\Delta^2 W - \rho_1 h_1 \omega^2 W + g\rho W - i\omega\rho\Phi_1 = -P(x, y) \quad (1)$$

$$(x, y \in \Omega_1)$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the 2-D Laplacian, ρ_1 is the plate density, h_1 is the plate thickness, g is the acceleration due to gravity.

According to linear shallow-water theory, we have

$$W = -\frac{i(h-d)}{\omega}\Delta\Phi_1 \quad (x, y \in \Omega_1) \qquad (2)$$

$$W = \frac{i\omega}{g}\Phi_2 \quad (x, y \in \Omega_2)$$

where $d = \rho_1 h_1/\rho$ is the draft of the plate.

Since the plate is freely floating, the bending moment and shear force should vanish at the edges of the plate:

$$\Delta W = \nu_1 \frac{\partial^2 W}{\partial s^2}, \quad \frac{\partial \Delta W}{\partial n} = -\nu_1 \frac{\partial^3 W}{\partial n \partial s^2} \quad (x, y \in S) \tag{3}$$

where $\nu_1 = 1 - \nu$, n and s denote the normal and tangential directions, ν is Poisson's ratio. At the corners of the plate, there can be concentrated shear force to compensate for the torsional moment along the edges of the plate. The vanishing of this shear force leads to

$$\frac{\partial^2 W}{\partial x \partial y} = 0 \quad (x = \pm\frac{B}{2},\ y = \pm\frac{L}{2}) \tag{4}$$

In the free-water region, the potential $\Phi_2(x, y)$ satisfies the equation

$$\Delta\Phi_2 + k_0^2 \Phi_2 = 0 \quad (x, y \in \Omega_2), \quad k_0 = \omega/\sqrt{gh} \tag{5}$$

Along the juncture boundary S, the continuity of mass flux and depth-mean pressure leads to the following matching conditions:

$$\frac{\partial \Phi_1}{\partial n} = \frac{h}{h-d} \frac{\partial \Phi_2}{\partial n}, \quad \Phi_1 = \Phi_2 \quad (x, y \in S) \tag{6}$$

Far from the plate, the radiation condition has to be imposed

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} - ik_0\right)\Phi_2 = 0, \quad r = \sqrt{x^2 + y^2} \tag{7}$$

The solution of the boundary-value problem in region Ω_1 can be decomposed as

$$\Phi_1(x, y) = \Phi_0(x, y) + \sum_{m=1}^{3} \Psi_m(x, y)$$

Here the potential $\Phi_0(x, y)$ is the solution of the formulated problem in the case of infinite elastic plate. This solution can be obtained by integral Fourier transforms (see *e.g.* Cherkesov, 1973). The functions $\Psi_m(x, y)$ ($m = 1, 2, 3$) satisfy the following equation

$$\Delta\Psi_m + \mu_m^2 \Psi_m = 0 \tag{8}$$

where μ_m ($m = 1, 2, 3$) are the roots of the equation

$$\frac{D}{\rho}\mu^6 + (g - d\omega^2)\mu^2 - \frac{\omega^2}{h-d} = 0 \tag{9}$$

The positive real root of this equation is denoted by μ_1, and two complex-conjugate roots located in the first and the fourth quadrants of the complex plane are denoted by μ_2 and μ_3, respectively.

The Helmholtz-type equations (5) and (8) are solved by the boundary-integral-equation method involving the Green functions, and the finite-difference method is used for discretization of the boundary conditions (3) and (4) at the edges of the plate. This method has been used earlier for the study of the hydroelastic response of a mat-type floating runway in regular, oblique waves by Ertekin & Kim (1999) and by Sturova (2001).

Trapped modes of an elastic floating strip

The existence of trapped mode solution for an elastic strip floating on shallow water was shown by Tkacheva (2000). Let us briefly consider the solution of this problem. The elastic strip with the finite width B and infinite length is considered. Trapped modes are nontrivial solutions of the homogeneous boundary-value problem, formulated as consequence of Eqs. (1), (2) and (5)

$$\frac{D}{\rho}\Delta^3\Phi_1 + (g - d\omega^2)\Delta\Phi_1 + \frac{\omega^2}{h-d}\Phi_1 = 0 \tag{10}$$

$$(|x| \le \frac{B}{2},\ |y| < \infty)$$

$$\Delta\Phi_2 + k_0^2\Phi_2 = 0 \quad (|x| > \frac{B}{2},\ |y| < \infty) \tag{11}$$

with the free-edge conditions that follow from (2) and (3)

$$\Delta^2\Phi_1 = \nu_1\Delta\frac{\partial^2\Phi_1}{\partial y^2}, \quad \Delta^2\frac{\partial\Phi_1}{\partial x} = -\nu_1\Delta\frac{\partial^3\Phi_1}{\partial x \partial y^2} \tag{12}$$

$$(x = \pm\frac{B}{2},\ |y| < \infty)$$

and the matching conditions resulting from (6)

$$\frac{\partial\Phi_1}{\partial x} = \frac{h}{h-d}\frac{\partial\Phi_2}{\partial x}, \quad \Phi_1 = \Phi_2 \quad (x = \pm\frac{B}{2},\ |y| < \infty) \tag{13}$$

The solution of Eqs. (10) and (11), corresponding to waves of wavenumber λ travelling along the strip, can be sought in the form

$$\Phi_j(x, y) = \Psi_j(x)\exp(i\lambda y), \quad j = 1, 2 \tag{14}$$

The mode is said to be trapped if

$$\Psi_2 \to 0 \quad (|x| \to \infty)$$

Then the solution for $\Psi_2(x)$ can be written with the help of (11) and (14), as:

$$\Psi_2(x) = \alpha_+ \exp(-\beta x) \quad (x > B/2)$$

$$\Psi_2(x) = \alpha_- \exp(\beta x) \quad (x < -B/2)$$

where $\beta^2 = \lambda^2 - k_0^2$, and $\alpha_\pm$ are unknown constants. The value of β should be real and positive, consequently $\lambda > k_0$. This inequality implies that the trapped modes in the floating elastic strip cannot be excited by incoming waves, because for incoming wave inequality $\lambda \le k_0$ is always valid (see *e.g.* Sturova, 1998).

Trapped modes in the floating elastic strip exist only at nonzero draft of the strip. They are represented by the even function of x (Tkacheva, 2000). The solution for $\Psi_1(x)$ can be written in the form

$$\Psi_1(x) = \sum_{m=1}^{3} c_m \cosh(\sigma_m x) \quad (|x| \leq B/2) \qquad (15)$$

where c_m are unknown constants and the values σ_m are determined from the equation similar to (9) upon the substitution of (14) in (10) with due regard to (15). From the free-edge conditions (12) and the matching conditions (13) we obtain the system of linear algebraic equations of the fourth order. Trapped modes correspond to the vanishing of the determinant of this system.

A special feature of trapped modes in the floating elastic strip is that for each $d > 0$ they exist only in the region of the frequencies $0 < \omega < \omega_{cr}$. The critical value ω_{cr} increases with the draft.

3. A BOTTOM TOPOGRAPHY

Let h_1 be the depth of the fluid inside a region Ω_1 bounded by the contour S and h_2 be the depth of the remaining fluid Ω_2. We restrict ourselves to the case where the localized external pressure $P(x, y)$ is located inside the region Ω_1. This problem can be considered as the particular case of the previous problem.

According to the linear shallow-water theory, the velocity potentials $\Phi_j(x, y)$ $(j = 1, 2)$ are found by solving the system of equations

$$\Delta\Phi_1 + k_1^2\Phi_1 = -\frac{i\omega}{g\rho h_1} P(x, y) \quad (x, y \in \Omega_1) \qquad (16)$$

$$\Delta\Phi_2 + k_2^2\Phi_2 = 0 \quad (x, y \in \Omega_2) \qquad (17)$$

with matching conditions along the boundary S

$$\gamma\frac{\partial\Phi_1}{\partial n} = \frac{\partial\Phi_2}{\partial n}, \quad \Phi_1 = \Phi_2 \quad (x, y \in S) \qquad (18)$$

where $\gamma = h_1/h_2$, $k_j = \omega/\sqrt{gh_j}$ $\quad (j = 1, 2)$.

In the far field $\Phi_2(x, y)$ must satisfy the radiation condition by analogy with (7).

The solution of Eq. (16) is sought in the form

$$\Phi_1(x, y) = \Phi_0(x, y) + \Psi(x, y)$$

where $\Phi_0(x, y)$ is the solution of the formulated problem in the case of a fluid with a constant depth h_1 and $\Psi(x, y)$ has to be determined. The function $\Phi_0(x, y)$ is found by integral Fourier transform (see *e.g.* Cherkesov, 1973). The function $\Psi(x, y)$ satisfies the equation

$$\Delta\Psi + k_1^2\Psi = 0 \quad (x, y \in \Omega_1) \qquad (19)$$

The solution of the Helmholtz-type equations (17) and (19) can be obtained by the method of integral equations, which has been used to study surface wave diffraction on a rectangular pit (Williams, 1990).

Trapped modes of a rectangular ridge

Let us consider the bottom topography of uniform width B and infinite length. The depth of the fluid in the domain $|x| < B/2$, $|y| < \infty$ is equal to h_1, and the depth of fluid is equal to h_2 in the remaining part.

To determine the trapped modes for this topography, we have to find a nontrivial solution of the homogeneous equations for the corresponding velocity potentials

$$\Delta\Phi_1 + k_1^2\Phi_1 = 0 \quad (|x| < B/2,\ |y| < \infty) \qquad (20)$$

$$\Delta\Phi_2 + k_2^2\Phi_2 = 0 \quad (|x| > B/2,\ |y| < \infty) \qquad (21)$$

with matching conditions on the straight lines $|x| = B/2$ similar to (18).

The solutions of Eqs.(20) and (21) are sought in the form

$$\Phi_j(x, y, t) = \Psi_j(x) \exp(i\lambda y), \quad j = 1, 2$$

In the far field for trapped modes we have

$$\Psi_2 \to 0 \quad (|x| \to \infty)$$

The net result is that the trapped modes should satisfy the following equations:
for symmetric modes

$$\tanh\frac{\sigma B}{2} = -\frac{\beta}{\gamma\sigma} \qquad (22)$$

for antisymmetric modes

$$\tanh\frac{\sigma B}{2} = -\frac{\gamma\sigma}{\beta}. \qquad (23)$$

where $\sigma^2 = \lambda^2 - k_1^2$, $\beta^2 = \lambda^2 - k_2^2$. Eqs. (22) and (23) can have real roots only for $\gamma < 1$ $(h_1 < h_2)$ in the frequency range $\lambda\sqrt{gh_1} < \omega < \lambda\sqrt{gh_2}$. Hence, only ridge-type topography possesses waveguide properties. The trapped modes exist for any frequency in contrast to the first problem. The number of modes increases indefinitely with increasing frequency, which is a feature of the shallow-water approximation.

4. NUMERICAL RESULTS

The isolines of amplitudes of the deflections of the elastic plate or the elevations of the water surface, as well as the scattering diagram for the surface waves in far field, will be presented at the Workshop. An example is given in Fig. 2 which shows the scattering diagram for the floating elastic plate.

The material properties, water depth and the main dimensions are taken as:

$$D = 1.96 \cdot 10^{11}\ N \cdot m,\ \nu = 0.3,\ h = 50\ m,$$

$$B = 1\ km,\ d = 5\ m$$

The critical frequency for the trapped mode in this elastic strip of infinite length is equal to $\omega_{cr} \approx 0.199 s^{-1}$ ($\omega_{cr}\sqrt{h/g} \approx 0.450$).

Two values of the plate length are considered:

$$L = 5\ km \quad \text{(case 1)} \quad \text{and} \quad L = 7\ km \quad \text{(case 2)}$$

The first case corresponds to the project of the floating runway (Ertekin & Kim, 1999).

The distribution of the external pressure in (1) has the form

$$P(x, y) = ag\rho f(R)$$

Here a is a coefficient with a dimensionality of length, the function $f(R)$ is chosen as

$$f(R) = 1 - (R/l)^2 \quad (R < l), \quad f(R) = 0 \quad (R > l)$$

where $R = \sqrt{(x - x_0)^2 + (y - y_0)^2}$, x_0 and y_0 are the coordinates of the epicenter of the external pressure region.

It is known, that in the considered problems the amplitudes of surface waves in the far field decrease with increasing distance from the origin of coordinates r as $r^{-1/2}$. Then the amplitudes of surface waves in the far field can be written as

$$\frac{|W|}{a} = \frac{\omega h^2}{(h - d)\sqrt{gr}}\, Q(\theta) \quad (r \to \infty)$$

where $\theta = \arctan(y/x)$.

The behavior of the dimensionless function $Q(\theta)$ is shown in polar coordinates in Fig. 2 for $l = 100\ m$. The left part of this figure corresponds to the plate with $L/B = 5$ ($x_0 = 0,\ y_0 = -1250\ m$), and the right one – to $L/B = 7$ ($x_0 = 0,\ y_0 = -2250\ m$). The distance of the epicenter of the pressure region from the plate edge $y = -L/2$ is equal to 1250 m in both cases. The solid, dashed and dot-and-dashed curves in Fig. 2 correspond to the subcritical frequency $\omega\sqrt{h/g} = 0.3$, the critical frequency $\omega\sqrt{h/g} = 0.45$ and the supercritical frequency $\omega\sqrt{h/g} = 0.6$, respectively.

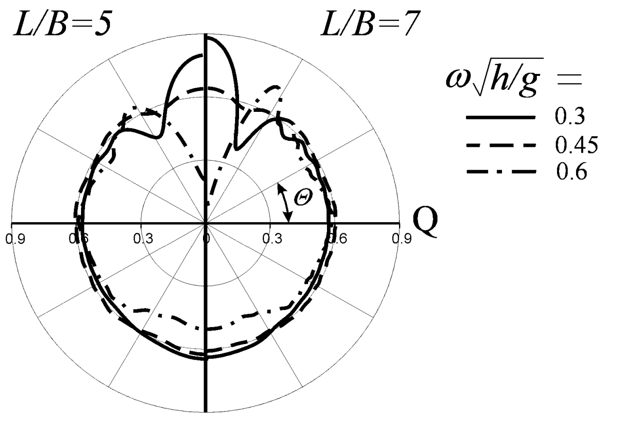

Figure 2: Scattering diagram $Q(\theta)$

From this figure we notice that the shape of scattering diagram essentially differs at $\omega < \omega_{cr}$ and $\omega > \omega_{cr}$. Trapped modes in elastic strip exist only for subcritical frequencies. Waveguide property can be manifested at $\omega < \omega_{cr}$: the surface waves with the largest amplitude take place at $\theta = \pi/2$, that is along the positive y-axis. This effect increases with the aspect ratio of the plate. At the supercritical frequency the surface wave amplitude is minimum at $\theta = \pi/2$, and the surface waves spread in the transversal direction. At the critical frequency the scattering diagram has nearly circular shape.

5. CONCLUSION

The behavior of waves generated by periodic pressure is considered within the linear shallow-water theory. An effective method for investigating the dynamics of an elastic floating platform of arbitrary shape is proposed. A particular case of this problem is considered also: a bottom topography with a piecewise-constant fluid depth. Manifestation of waveguide properties of the elongated rectangular structures is shown.

ACKNOWLEDGMENT

The research was supported by the grant of President of Russian Federation for Leading Scientific Schools (NSH-902.2003.1).

REFERENCES

Cherkesov, L.V. (1973) *Surface and Internal Waves*, Naukova Dumka, Kiev, (in Russian).

Ertekin, R.C. & Kim, J.W. (1999) Hydroelastic response of a floating mat-type structure in oblique, shallow-watre waves. *J. Ship Res.*, **43(4)**, pp. 241-254.

Evans, D.V. & Kuznetsov, N. (1997) Trapped modes. *Gravity Waves in Water of Finite Depth,* J.N.Hunt, ed., Computational Mechanics Publication, Southampton, U.K., pp. 127-168.

Le Blond, P.H. & Mysak, L.A. (1978) *Waves in the Ocean.* Elsevier, Amsterdam.

Sturova, I.V. (1998) The oblique incidence of surface waves onto the elastic band. *Hydroelasticity in Marine Technology*, M.Kashiwagi et al., eds, Yomei Printing Cooperative Society, Fukuoka, Japan, pp. 239-245.

Sturova, I.V. (2001) The diffraction of surface waves by an elastic platform floating on shallow water. *J. Appl. Maths. Mechs.*, **65(1)**, pp. 109-117.

Tkacheva, L.A. (2000) Eigenvibrations of a flexible platform floating on shallow water. *J. Appl. Mechs. and Techn. Phys.*, **41(1)**, pp. 159-166.

Williams, A.N. (1990) Diffraction of long waves by rectangular pit. *J. Waterway, Port, Coastal, and Ocean Eng.*, **116(4)**, pp. 459-469.