Computations of three-dimensional gravity-capillary solitary waves in finite depth

Emilian I. Părău, Jean-Marc Vanden-Broeck, Mark J. Cooker
School of Mathematics, University of East Anglia, Norwich, NR4 7TJ, UK
e-mail: e.parau@uea.ac.uk

1 Introduction

The classical problem of three-dimensional water waves in finite depth when both gravity and surface tension are present is considered.

The dispersion relation for linearized capillary-gravity waves travelling at a constant velocity c on water of finite depth h can be written as (see for instance Lamb (1932))

$$D(\kappa;\lambda,\beta) \equiv (\lambda+\beta\kappa^2)\tanh\kappa-\kappa=0 \tag{1}$$

where $\kappa = k^*h$ is the dimensionless wavenumber. The problem involves two dimensionless numbers

$$\lambda=\frac{gh}{c^2} \quad \text{and} \quad \beta=\frac{T}{\rho h c^2}$$

where T is the constant coefficient of surface tension, g is the acceleration due to gravity, ρ is the fluid density. The parameter λ is the inverse of the square of the Froude number and both parameters are related to the Bond number $B = T/\rho g h^2$ by the relation $B = \beta/\lambda$.

Parau, Vanden-Broeck and Cooker (2004) computed fully nonlinear gravity capillary lumps in water of infinite depth. Their results are consistent with those of Kim and Akylas (2004) who showed analytically that lumps can bifurcate from linear sinusoidal waves with wavenumber corresponding to the minimum gravity-capillary phase which is also a double root of the dispersion relation (1). Three dimensional solitary waves were also obtained, for a weakly nonlinear model, by Milewski (2005).

Groves and Sun (2004) have shown that fully localised solitary waves also exist in the case $\beta > 1/3, \quad 0 < \lambda - 1 \ll 1$. In this region of strong surface tension Kadomtsev and Petviashvili (1970) derived the well-known KP-I equation as a long-wave approximation for solutions of the steady gravity-capillary water wave problem which has fully localised solitary-wave solutions.

In this paper we compute fully nonlinear three-dimensional gravity-capillary solitary waves of the full Euler equations on finite depth by a boundary integral equation method. For the case of strong surface tension we compare with solutions of the KP-I equation.

2 Formulation and numerical scheme

The fluid is incompressible and the flow is irrotational. We are only interested in steady waves which travel at a constant velocity c on water of finite depth h, and we choose a frame of reference moving with this wave speed. We introduce cartesian coordinates

x, y, z with the z-axis directed vertically upwards and the x-axis in the direction of the wave motion. We denote by $z = \zeta(x, y)$ the equation of the free surface. Dimensionless variables are introduced by taking the unit length to be $T/\rho c^2$ and the unit velocity to be c. In terms of the velocity potential function $\Phi(x, y, z)$, the problem is formulated as follows:

$$\nabla^2\Phi = 0, \qquad x, y \in \mathbf{R}, z < \zeta(x, y), \tag{2}$$

with the boundary conditions

$$\Phi_x\zeta_x + \Phi_y\zeta_y = \Phi_z, \quad \text{on } z = \zeta(x, y), \tag{3}$$

$$\frac{1}{2}(\Phi_x^2 + \Phi_y^2 + \Phi_z^2) + \lambda\beta\zeta - \left[\frac{\zeta_x}{\sqrt{1 + \zeta_x^2 + \zeta_y^2}}\right]_x - \left[\frac{\zeta_y}{\sqrt{1 + \zeta_x^2 + \zeta_y^2}}\right]_y = \frac{1}{2}, \text{ on } z = \zeta(x, y), \tag{4}$$

$$\Phi_z = 0, \quad \text{on} \quad z = -\frac{1}{\beta}. \tag{5}$$

As we are looking for fully localised three-dimensional solitary waves, we have used the conditions

$$(\Phi_x, \Phi_y, \Phi_z) \to (1, 0, 0) \quad \text{and} \quad \zeta \to 0, \text{ as } \quad (x^2 + y^2)^{1/2} \to \infty. \tag{6}$$

to fix the value of Bernoulli's constant in the equation (4).

The numerical procedure is an extension to finite depth of the scheme used by Forbes (1989) and by Părău and Vanden-Broeck (2002) for pure gravity waves, and by Părău *et al* (2004) for the computation of the fully localised gravity-capillary waves in deep water. It is based on a boundary integral equation method introduced by Forbes (1989) for three dimensional gravity free surface flows due to moving pressure distributions.

The formulation involves applying Green's second identity to the functions $\Gamma = \Phi - x$ and G the three dimensional free space Green function

$$G = \frac{1}{4\pi}\frac{1}{((x - x^*)^2 + (y - y^*)^2 + (z - z^*)^2)^{1/2}}, \tag{7}$$

for a volume V which consists of a cylinder bounded by the free surface S_F (except a small hemisphere around the point $Q(x^*, y^*, z^*)$), and its image $S_{F'}$ on the other side of the bottom $z = -1/\beta$. In that way, by symmetry, the condition of no flow normal to the bottom (5) is satisfied.

After projecting the surface integrals onto the Oxy plane, we obtain

$$\frac{1}{2}(\phi(x^*, y^*) - x^*) =$$

$$= \int\int_{\mathbf{R}^2}(\phi(x, y) - x)\frac{1}{4\pi}\frac{\zeta(x, y) - \zeta(x^*, y^*) - (x - x^*)\zeta_x(x, y) - (y - y^*)\zeta_y(x, y)}{((x - x^*)^2 + (y - y^*)^2 + (\zeta(x, y) - \zeta(x^*, y^*))^2)^{3/2}}dxdy+$$

$$+\int\int_{\mathbf{R}^2}\frac{1}{4\pi}\frac{\zeta_x(x, y)}{((x - x^*)^2 + (y - y^*)^2 + (\zeta(x, y) - \zeta(x^*, y^*))^2)^{1/2}}dxdy+$$

$$+\int\int_{\mathbf{R}^2}(\phi(x,y)-x)\frac{1}{4\pi}\frac{\zeta(x,y)+\zeta(x^*,y^*)+2/\beta-(x-x^*)\zeta_x(x,y)-(y-y^*)\zeta_y(x,y)}{((x-x^*)^2+(y-y^*)^2+(\zeta(x,y)+\zeta(x^*,y^*)+2/\beta)^2)^{3/2}}dxdy+$$

$$+\int\int_{\mathbf{R}^2}\frac{1}{4\pi}\frac{\zeta_x(x,y)}{((x-x^*)^2+(y-y^*)^2+(\zeta(x,y)+\zeta(x^*,y^*)+2/\beta)^2)^{1/2}}dxdy \tag{8}$$

where $\phi(x,y)=\Phi(x,y,\zeta(x,y))$.

The singularities on the integrals may be isolated in one term by addition and substraction of a null quantity, which can be evaluated in closed form (see Forbes (1989) for details). One different feature of our numerical scheme is that second derivatives now appear in the curvature term in (4). They are approximated by centred difference formulas. A new feature is that no radiation condition is needed. Instead the solutions are assumed to be symmetric about the x and y axes. The discretization involves truncation and a regular grid with N points in the x direction and M in the y direction. The algebraic equations obtained after discretization are solved by Newton's method.

The numerical algorithm can be extended to calculate waves which propagate along the interface between two superposed fluid layers of finite or infinite thickness.

3 Results

For small surface tension ($\beta < 1/3$) we found that the three dimensional problem is qualitatively similar to the two dimensional problem. In particular there are two branches of fully localised central depression, or central elevation, three dimensional gravity capillary solitary waves. These waves have decaying oscillations in the direction of propagation and are monotonically decaying perpendicular to the direction of propagation (see Fig.1(a)). The curves obtained by cutting the free surface with planes parallel to the direction of propagation are qualitatively similar to the two dimensional profiles obtained by Vanden-Broeck and Dias (1992) and by Dias *et al.* (1996). The solutions are quite similar to the fully localised solitary waves found on deep water (see Parau *et al.* 2004).

For strong surface tension ($\beta > 1/3$) we found only fully localised depression gravity capillary solitary waves. They are similar to fully localised solitary-wave solutions of the KP-I equation as shown in Fig. 1(b). The amplitude of the solutions is also close to that predicted by KP-I for β bigger than 1/3 and λ close to 1. We can follow continuously a branch of central depression solitary-wave solutions from strong to weak surface tension, by passing through $\beta = 1/3$, as predicted by Milewski (2005). There is a maximum amplitude for the branch of central depression waves (with λ constant) for β near 1/3.

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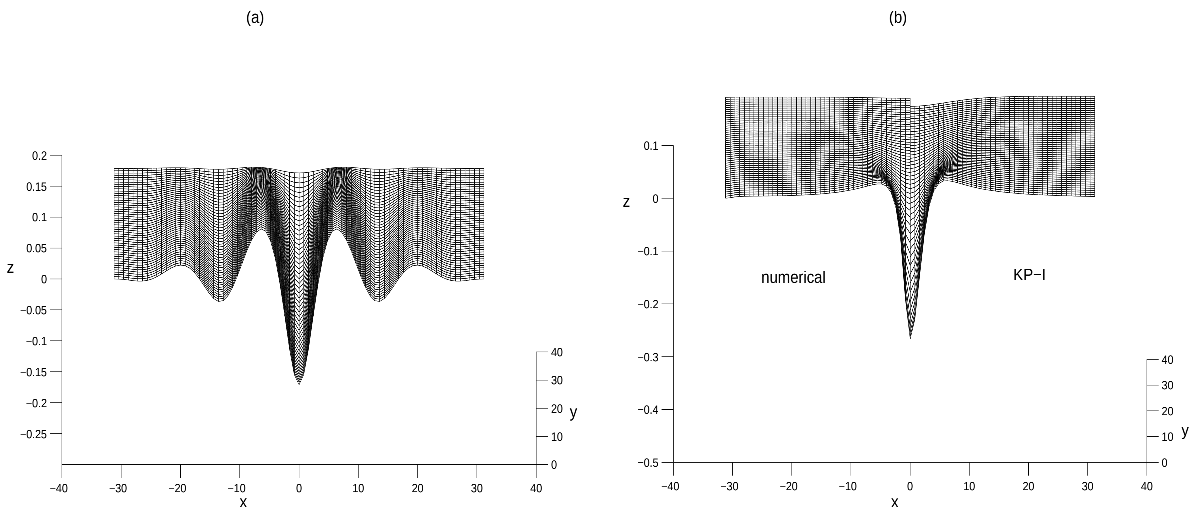

Figure 1: (a) Solitary gravity-capillary wave for $\beta = 0.25, \lambda = 1.13$.
(b) Solitary gravity-capillary wave for $\beta = 1.2, \lambda = 1.13$ ($x \leq 0$) compared with the fully localised solitary wave solution for KP-I ($x \geq 0$), as given by Milewski (2005). Only half of the solutions ($y \geq 0$) are shown.

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