Trapped modes for surface-piercing cylinders below and above the cut-off frequency.

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1. Introduction and statement of the problem

In the present paper existence of trapped modes is proved both below and above the cut-off frequency in the linear problem, which describes interaction of waves with surface-piercing cylindrical bodies spanning vertical walls in a channel of a finite width and infinite depth. Above the cut-off the results are also applicable to the oblique waves problem. The existence of surface-piercing bodies supporting trapped modes below the cut-off was not earlier proved for the problem under consideration.

Let an inviscid, incompressible fluid with a free surface contain a system of horizontal, partly submerged cylinders of uniform cross-section. Cartesian coordinates are chosen with the z-axis directed along the generators of the cylindrical geometry, the origin lying in the free surface and y measured vertically upwards, so that the fluid in absence of bodies occupies $y \leqslant 0$. Then on the plane (x, y): W denotes the cross-section of the domain occupied by fluid, S is the wetted surface of bodies and F is the free surface.

We seek a solution to the homogeneous problem

$$\Delta u - k^2 u = 0 \quad \text{in} \quad W, \quad u_y - \nu u = 0 \quad \text{on} \quad F, \quad \partial u/\partial n = 0 \quad \text{on} \quad S, \tag{1}$$

$$\int_W |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}y + \nu \int_F |u|^2 \, \mathrm{d}x < \infty. \tag{2}$$

which describes trapped modes, i.e. unforced localized oscillations of fluid. The corresponding value of ν is a point eigenvalue, which is embedded into continuous spectrum when $\nu > k$. In the latter case the set of conditions allows propagation of waves to infinity while below the cut-off ($\nu < k$) solutions decays at infinity. Derivation of the problem both for the channel of width $2b$, when $kb = n\pi$ or $kb = (2n-1)\pi/2$, $n = 1, 2, \ldots$, and for the fluid unbounded in horizontal directions in the case of arbitrary angle θ between crests of incident waves and the plane normal to the generators of the cylinders, when $k = \nu \sin\theta < \nu$, can be found e.g. in [3].

Our purpose is to prove that *for all values ν and k, such that $\nu \neq k$, there exists a system of surface-piercing bodies S_i, $i = 1, 2, \ldots, N$, for which the problem* (1)–(2) *has non-trivial solution. Here $N \geqslant 2$ for $\nu > k$ and $N \geqslant 1$ for $\nu < k$.* In order to construct the trapped modes we use dipole modification [5] of the so-called inverse method suggested by M. McIver [4] for $k = 0$, for simultaneous construction of bodies and potential of a mode with finite energy trapped by these bodies. For this aim a potential combined from singularities located at the free surface is used and field (tangential to the flow) lines enclosing the points are sought to be chosen as contours of bodies.

Main features of the problem (1)–(2) in comparison with that in [4] ($k = 0$) are the existence of the cut-off frequency $k = \nu$ and the fact that its solution is not a harmonic function. The latter makes it impossible to define complex conjugate for the potential and the field lines are described as trajectories of a system of differential equations. This creates essential difficulty in proving existence of lines enclosing singularities.

First trapped modes for the problem under consideration for frequencies embedded into the continuous spectrum ($\nu > k$) were given in [3], but existence of the trapped modes, which are delivered by potentials composed from sources situated in the free surface, was proved only provided that k is small enough. Following [5] it is possible to overcome the restriction by using potentials composed from x-derivatives of sources. Unlike the potentials in [3], field lines of the dipole, which have both ends in the free surface, enclose singularity and can be fixed as body's contour, exist in any small vicinity of the dipole. The latter allows us to apply asymptotic methods to prove existence of trapped modes.

It is important that the approach is also applicable below the cut-off. Existence of localized solutions when $\nu < k$ is well established [1,6,8]. However, it is only proved for totally submerged bodies and existence of surface-piercing bodies supporting trapped modes is a new result. In [3] it was proved that there are no trapped modes below cut-off for surface-piercing bodies satisfying the so-called John's geometrical condition (any vertical line emanating from the free surface does not intersect wetted surface of the body). The configurations found here do not satisfy the condition and show importance of additional geometrical conditions in uniqueness theorems.

2. Construction of the trapped mode potential

We define the potential by placing one (below the cut-off) or two (above the cut-off) dipoles in the free surface as follows

$$\Phi(x,y) = -\frac{\pi}{\nu}\big[G_x(x,y,0) - H(\nu-k)\,G_x(x,y,-2a)\big], \quad a = \pi\big(\nu^2-k^2\big)^{-1/2}, \tag{3}$$

where $G(x,y,\xi,\eta)$ is the potential of source located at the point (ξ,η) (see [3,7]), H is the Heaviside function.

Using known expression for the Green function we find

$$\Phi(x,y) = \frac{1}{\nu}\int_0^\infty \frac{t\mathrm{e}^{y\sqrt{t^2+k^2}}}{\sqrt{t^2+k^2}-\nu}\big[\sin xt - H(\nu-k)\sin(x+2a)t\big]\,\mathrm{d}t,$$

and the expression under the integral sign has no singularities (above the cut-off it is due to the special choice of distance between the singularities). The potential (3) satisfies first two conditions (1) by definition of Green's function. Besides, the potential satisfies the condition (2) in any domain W of lower half-plane excluding vicinities of the singular points.

If the field lines of the potential Φ are given by $(x(s),y(s))$, they satisfies the following system of differential equations

$$\dot{x} = \Phi_x(x(s),y(s)), \quad \dot{y} = \Phi_y(x(s),y(s)), \quad \text{where} \quad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}s}, \quad \dot{y} = \frac{\mathrm{d}y}{\mathrm{d}s}. \tag{4}$$

A cylinder cross-section having such a field line as a boundary, with local normal $\boldsymbol{n}$, satisfies $\nabla\Phi\cdot\boldsymbol{n} = 0$, i.e. the third condition (1).

Above the cut-off, where we have two singular points, it is to note that $\Phi(x-a,y)$ is symmetric in x, the picture of field lines is symmetric with respect to the line $\{x=-a\}$ and the field lines can only be considered in a vicinity of the right dipole situated at the origin. So, we have to establish existence of trajectories of (4) with both ends in the free surface, enclosing the singularity of the potential located at the origin.

By an algebra we have $\Phi(x,y) = x\nu^{-1}(x^2+y^2)^{-1} + \arg\nu(x+\mathrm{i}y) + \phi(x,y)$, where $\phi(x,y)$ is a function which is continuous at the origin. The sum of the first two terms in the representation of Φ is equal to $\mathrm{Re}\{w(\nu(x+\mathrm{i}y))\}$, where $w(\nu(x+\mathrm{i}y) = 1/\nu(x+\mathrm{i}y) - \mathrm{i}\log\nu(x+\mathrm{i}y)$ and we could suppose that near the singular point the trajectories of (4) are close to the streamlines $\mathrm{Im}\{w(\nu(x+\mathrm{i}y)\} = \mathrm{const}$ of the potential w and have similar properties enclosing the singular point in any small vicinity of the origin (unlike the potentials composed from sources, which were used in [3]).

3. Existence of field lines enclosing the origin inside

We find the following asymptotic representations as $r = |x + \mathrm{i}y| \to 0$

$$\begin{gathered}
\Phi_x = \frac{y^2 - x^2}{\nu\,(x^2+y^2)^2} - \frac{y}{x^2+y^2} + O(\log r), \qquad \Phi_y = -\frac{2xy}{\nu\,(x^2+y^2)^2} + \frac{x}{x^2+y^2} + O(1), \\
\Phi_{xy} = \frac{2y\,(3x^2-y^2)}{\nu\,(x^2+y^2)^3} + O\big(r^{-2}\big), \quad \Phi_{xx} = \frac{2x\,(x^2-3y^2)}{\nu\,(x^2+y^2)^3} + O\big(r^{-2}\big),
\end{gathered} \tag{5}$$

and the above asymptotics for Φ_{xx} is also valid for Φ_{yy}.

It is convenient to change parametrization in (4), rewriting the system in the form

$$\dot{x} = X(x(s), y(s)), \quad \dot{y} = Y(x(s), y(s)) \tag{6}$$

where $X(x,y) = \nu\,\big(x^2+y^2\big)^{3/2}\,\Phi_x(x,y)$ and $Y(x,y) = \nu\,\big(x^2+y^2\big)^{3/2}\,\Phi_y(x,y)$. Besides, we consider the model system with the right-hand sides containing principal terms of X and Y as $r \to 0$

$$\dot{x}_0 = X_0(x_0(s), y_0(s)), \quad \dot{y}_0 = Y_0(x_0(s), y_0(s)) \tag{7}$$

where $X_0(x,y) = \big(y^2 - x^2\big)\big(x^2+y^2\big)^{-1/2}$ and $Y_0(x,y) = -2xy\,\big(x^2+y^2\big)^{-1/2}$. It is easy to find that trajectories of (7) in $\overline{\mathbb{R}^2_-} = \{y \leqslant 0\}$ are circles. We use their parametrization $x_0(s) = \rho \sin\tau(s)$, $y_0(s) = -\rho(1 + \cos\tau(s))$, such that $\tau = \pm\pi$ correspond to the origin and $\tau(s) = \pm\pi + O(\mathrm{e}^{\mp s})$ as $s \to \pm\infty$.

Let $(x^\rho(s), y^\rho(s))$ and $(x_0^\rho(s), y_0^\rho(s))$ be the trajectories of (6) and (7) respectively, passing through the point $(0, -\rho)$, where $s = 0$. In view of (6), (7) and (5) maximal values of $\dot{x}^\rho(s)$, $\dot{y}^\rho(s)$, $\dot{x}_0^\rho(s)$ and $\dot{y}_0^\rho(s)$ in the semicircle $B_{d\rho} = \big\{(x,y) : |x + \mathrm{i}y| \leqslant d\rho,\ y \leqslant 0\big\}$ have the order $O(\rho)$ as $\rho \to 0$. Thus, for any $s_* < \infty$ we can find $d(c) \geqslant 1$ such that when $|s| \leqslant s_*$, $0 < \rho \leqslant c$, the semicircle $B_{d\rho}$ contains the trajectories $(x^\rho(s), y^\rho(s))$ and $(x_0^\rho(s), y_0^\rho(s))$. Note that $|\nabla X|$, $|\nabla Y|$, $|\nabla X_0|$ and $|\nabla Y_0|$ are bounded in vicinity of the origin and define

$$M(\rho) = \max\{|\nabla X_0|, |\nabla Y_0| : (x,y) \in B_{d\rho}\}, \qquad \varepsilon(\rho) = \max\{|X - X_0|, |Y - Y_0| : (x,y) \in B_{d\rho}\}$$

Then, the comparison theorem (Theorem 3.3.1, [2]) and the estimate $\varepsilon(\rho) = O\big(\rho^2\big)$ as $\rho \to 0$ guarantee that

$$\max\{\|\boldsymbol{x}^\rho(s) - \boldsymbol{x}_0^\rho(s)\| : |s| \leqslant s_*\} \leqslant \varepsilon(\rho)\big(\mathrm{e}^{M(\rho)s_*} - 1\big)\ /\ M(\rho) = O\big(\rho^2\big). \tag{8}$$

Consider a vicinity of the origin $V \subset \overline{\mathbb{R}^2_-}$, containing parts of the free surface both for $x > 0$ and $x < 0$. Let the vicinity be small so that the origin is the single stationary point ($X = Y = 0$) in V (existence of such a vicinity follows from (5)). We consider a set $\mathcal{S} \subset V$, formed by the solutions of (6), transversing the negative part of y-axis. Let $\mathcal{F}_\pm \subset V \setminus \mathcal{S}$ include parts of the free surface for $\pm x > 0$, respectively.

We should prove that the hypothesis that $\mathcal{S}$ has no common point with the free surface for $x < 0$, i.e. that the trajectory of (6) γ_*, separating $\mathcal{F}_-$ and $\mathcal{S}$ enters to the origin, leads to a contradiction. First we observe that the points $(x_0^\rho(s_*), y_0^\rho(s_*))$ belongs to a ray emanating from the origin. The angle between the ray and the free surface can be done as small as we wish by increasing the value s_*. Thus, in view of (8), γ_* is tangential to the free surface. Let $y_*(x)$ be the function which defines γ_* near the origin. Since $y_*(x) = o(x)$ as $x \to 0$, the formulae (6) and (5) implies that $y_*'(x)$ satisfies the equation $y_*'(x) = -\nu x + 2x^{-1} y_*(x) + \lambda(x), \quad \lambda(x) = o(x)$.

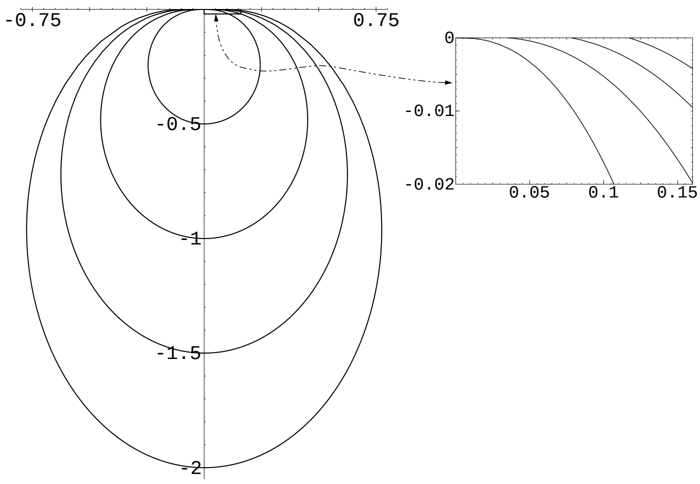

Figure 1.

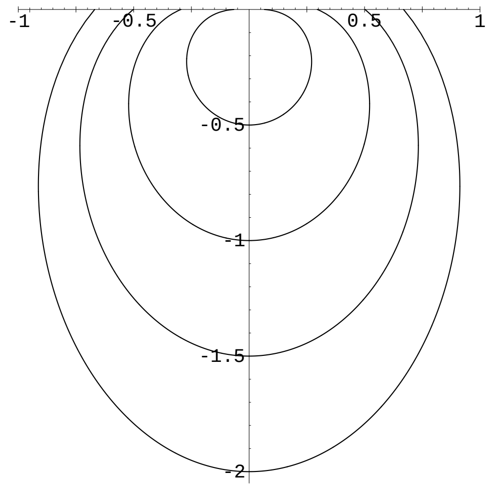

Figure 2.

An asymptotic analysis leads to the representation $y_*(x) = -\nu x^2 \log \nu|x| + o\big(x^2 \log x\big)$ as $x \to 0$. The latter contradicts to the above hypothesis because the line γ_* emanates from the origin to the upper half-plane. Thus, a family of trajectories $\mathcal{S}_0 \subset \mathcal{S}$ can be found having both ends in the free surface and enclosing the dipole at the coordinate origin and this is true for arbitrary k, ν, $k \neq \nu$.

Due to its local nature the proof can immediately be generalized for the problem (1)–(2) for a system consisting of N surface-piercing bodies, where $N > 1$ below the cut-off and $N > 2$ above the cut-off. Below the cut-off the potential can be constructed from dipoles located at arbitrary points of the free surface. Above the cut-off waves at infinity from the dipole solutions should be removed by a special choice of distance between singularities.

Shown in fig. 1, 2 are results of numerical computations of field lines (solutions of (4)) on the plane (x, y), which deliver configurations supporting trapped modes for $k = 1$, $\nu = 0.5$ (fig. 1) and $\nu = 0.95$ (fig. 2). A part of fig. 1 near the coordinate system origin is given in a different scale.

4. Conclusion

The inverse method [4] and its dipole modification [5] are developed to construct trapped modes for the linearized problem describing interaction of fluid with a system of surface-piercing cylinders. Existence of the trapped modes is proved for any values of parameters k, ν, such that $k \neq \nu$. To author's knowledge, existence of trapped modes below cut-off ($\nu < k$) for surface-piercing bodies has not been established earlier. The obtained results also demonstrate importance of additional geometrical conditions in the uniqueness theorems (see e.g. [3]).

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