Second-order Diffraction and Refraction of Water Waves

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1 Introduction

In the linearized approximation, the problem of diffraction by bodies of finite size in a sea of constant depth can now be treated by a variety of methods. Refraction by slowly varying depth can also be handled by the classical ray approximation. For combined refraction and diffraction, the mild-slope approximation (MSE) by Berkhoff [1] is a powerful tool which reduces the computation of 3D problem involving both refraction and diffraction to 2D, and can be efficiently solved by, e.g., the hybrid element method of Chen & Mei [4]. The linearized MSE has been extended by Chamberlain & Porter [2] to include both first- and second-order terms in the seabed slope. For still steeper bed slopes Massel [6] and Porter & Staziker [7] have shown that better accuracy can be achieved by including evanescent modes.

For waves of finite amplitude, the second-order theory for a simple vertical cylinder on a sea of constant depth has been a challenge to theoreticians for a long time. A complete theory was carried out only in 1989 by Kim and Yue [5], using the boundary integral equation method and in 1992 by Chau and Eatok-Taylor [3], using eigenfunction expansions and a three-dimensional Green's function.

In this paper we report our recent progress on second-order diffraction of bodies on slowly varying depth, for monochromatic incident waves. The mild-slope equation is first extended to second order in wave steepness, and the method of solutions is then described. Sample results are then discussed. We shall first show that for the limiting case of diffraction by a vertical circular cylinder on a horizontal seabed, this method gives the same results as those of Kim & Yue [5] and Eatok-Taylor [3], hence is equivalent to their methods. New results involving a circular shoal, with and without a harbor behind a coastline, are presented. Further extensions for random waves are reported in a companion paper.

2 Second-order mild-slope approximation

By including only the propagating mode but the first two orders of bed slope ($O(\mu) = \nabla h/kh \ll 1$) the first-order result is of course the the extended MSE of Chamberlain & Porter [2]. The numerical solution can be carried out by the hybrid-element method of Chen & Mei [4].

At the second order in nonlinearity, it is now necessary to add evanescent modes. We let the second-order potential be of the form

$$\psi = -\frac{ig}{2\omega}\sum_{m=0}^{\infty}\xi_m\frac{\cos\kappa_m(z+h)}{\cos\kappa_m h}, \tag{2.1}$$

where κ_0 is the imaginary root, corresponding to the propagating mode, of

$$-4\omega^2 = g\kappa_m \tan\kappa_m h \tag{2.2}$$

while $\kappa_m, m = 1, 2, 3, ...$ are the real roots, corresponding to the evanescent modes. The first root κ_0 is imaginary, corresponding to the propagating mode. This expansion was used by Porter and Staziker [7] to extend the realm of the linearized theory for steeper bed slope. It is easy to find that ξ_m are governed by the coupled matrix PDE with forcing :

$$\sum_{\ell=0}^{\infty}\{\nabla\cdot(A_{m\ell}\nabla\xi_\ell) + B_{m\ell}\nabla h\cdot\nabla\xi_\ell + C_{m\ell}\xi_l\} = -2i\omega F_m \tag{2.3}$$

where F_m arises from quadratic products of the first order terms. In the special case of constant depth, the matrices on the left are diagonal and the equations are uncoupled.

3 Strategy of solution

For coastal waters where only the effects of local topography is of major concern, the far field is often approximated by an infinite (or semi-infinite) sea of constant depth. With this model, the hybrid element method originally devised for linearized wave problems [4], can be extended to the nonlinear problems here. Accordingly let us divide the fluid domain into two regions : the near field Ω_A in which the bathymetry and coastal boundary are complex, and the far field Ω_F where the depth is constant and the coastline straight. The two fields are separated by a semi circle ∂A of radius $r = a$. Within Ω_A, discrete solutions will be sought via two-dimensional finite elements. Within Ω_F, the solution will be represented analytically as eigen-function expansions. The unknown nodal coefficients in Ω_A and the expansion coefficients in Ω_F will be found together by Galerkin method, subject to the following matching conditions at $r = a$:

$$(\xi_l)_{\Omega_A} = (\xi_l)_{\Omega_F}\,, \qquad r = a \tag{3.4}$$

$$\left(\frac{\partial \xi_l}{\partial r}\right)_{\Omega_A} = \left(\frac{\partial \xi_l}{\partial r}\right)_{\Omega_F}, \qquad r = a \ \text{(on } C_A) \tag{3.5}$$

where $(.)_{\Omega_A}$ denotes the solution in Ω_A, $(.)_{\Omega_F}$ denotes the solution in Ω_F, and ∂A is the border line between Ω_A and Ω_F where $r = a$.

In the far field Ω_F of constant sea depth, the solution is obtained analytically as follows. First we decompose the forcing function into two parts

$$F = \mathcal{P} + \mathcal{S} \tag{3.6}$$

where $\mathcal{P}$ represents the terms arising from quadratic interaction of the first-order progressive waves (incident and reflected (from the coast)), and $\mathcal{S}$ represents terms from interactions of the first-order progressive waves and scattered waves as well as the self-intreraction of the scattered waves. By introducing the following decomposition,

$$(\xi_\ell)_{\Omega_F} = (\xi_\ell^P)_{\Omega_F} + (\xi_\ell^S)_{\Omega_F} + (\xi_\ell^f)_{\Omega_F}, \tag{3.7}$$

we first solve separately for $(\xi_\ell^P)_{\Omega_F}$ and $(\xi_\ell^S)_{\Omega_F}$ as the responses to the inhomogeneous equations with the forcing terms P and S respectively. For the latter two-dimensional Green's functions are used. Care is taken that the weak radiation condition is satisfied by $(\xi_\ell^S)_{\Omega_F}$ at infinity. The part $(\xi_\ell^f)_{\Omega_F}$ is then the free waves satisfying the homogeneous equation and inhomogeneous boundary condition on $r = a$ so as to ensure continuity of pressure and flux with the finite-element solution in the near-field.

4 Sample numerical results

As a benchmark for later numerical solutions involving variable depth, we have worked out a new analytical solution for a semi-circular peninsula along a straight coast and in constant depth. In this case no finite elements are needed and $\xi_l^f = 0$, for all l. Our theory is exact. Figure 1 shows the second-order contributions to the free-surface displacement along a circular peninsula at $r = a$, with 225° incidence. Variations of the dimensionless amplitudes of the zeroth and second harmonic, $\eta_{2,2}^{(0)}, \eta_{2,2}^{(1)}$ due to the first-order motion, $\eta_{2,2}^{(2)}$ due to Φ_2 as well as the sum of $\eta_{2,2}^{(1)} + \eta_{2,2}^{(2)}$, are shown in Figure 1.

For a full circular cylinder in an open sea, the exact solutions have been found before found numerically by Kim & Yue [5] using boundary integral equations, and analytically by Chau and Eatok Taylor [3]. As a check for the correctness of our theory, we have shown that in the limit of 0° angle of incidence, our solution

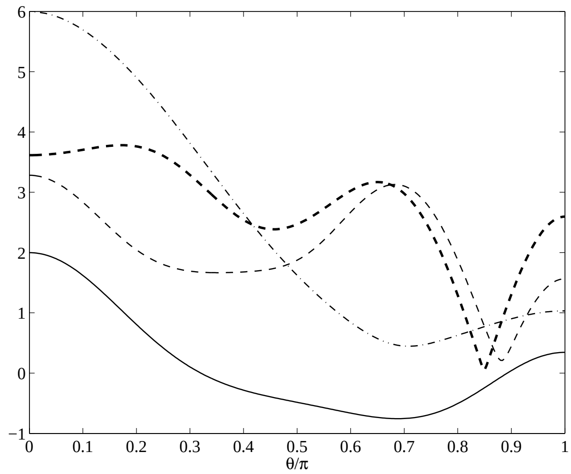

Figure 1: Dimensionless amplitudes of the second-order contribution to the free surface displacement along a circular peninsula with 225° incidence. Solid line : mean setup $\eta_{2,0}^{(1)}/(A^2/a)$ (zeroth harmonic component due to Φ_1). Chain line: second-harmonic component due to Φ_1, $|\eta_{2,2}^{(1)}|/(A^2/2a)$. Thin dashed line: second-harmonic component due to Φ_2, $|\eta_{2,2}^{(2)}|/(A^2/2a)$. Thick dashed line: sum of second-harmonic components due to $\Phi_1 + \Phi_2$, $|\eta_{2,2}^{(2)} + \eta_{2,2}^{(1)}|/(A^2/2a)$ .The input parameters are $r/a = 1$, $h/a = 1$ and $ka = 1$.

agrees with theirs. Unlike the 3D mathematics in these cited references, our treatment is an uncoupled set of 2D problems, a characteristic advantage of the mild-slope approximation.

In the second example we consider a semi circular shoal with one diameter coinciding with a straight coast. The depth profile of the shoal is given in polar coordinate by

$$h = \begin{cases} 20m, & r \leq 30\,m \\ 30 - 10\cos\left[\frac{\pi}{270}(r-30)\right]\, m, & 30\,m \leq r \leq 300\,m \end{cases} \tag{4.8}$$

Sample snapshots results for first-order and second-order surface displacements are shown in Figure 2 (a) and (b). For clarity, the two figures are shown in different scales.

Our last example is a square harbor open to a semi circular shoal. The incident wave frequency is chosen to coincide with a resonance mode. It happens that the secon harminc frequency is also close to the natural frequency of a higher mode.

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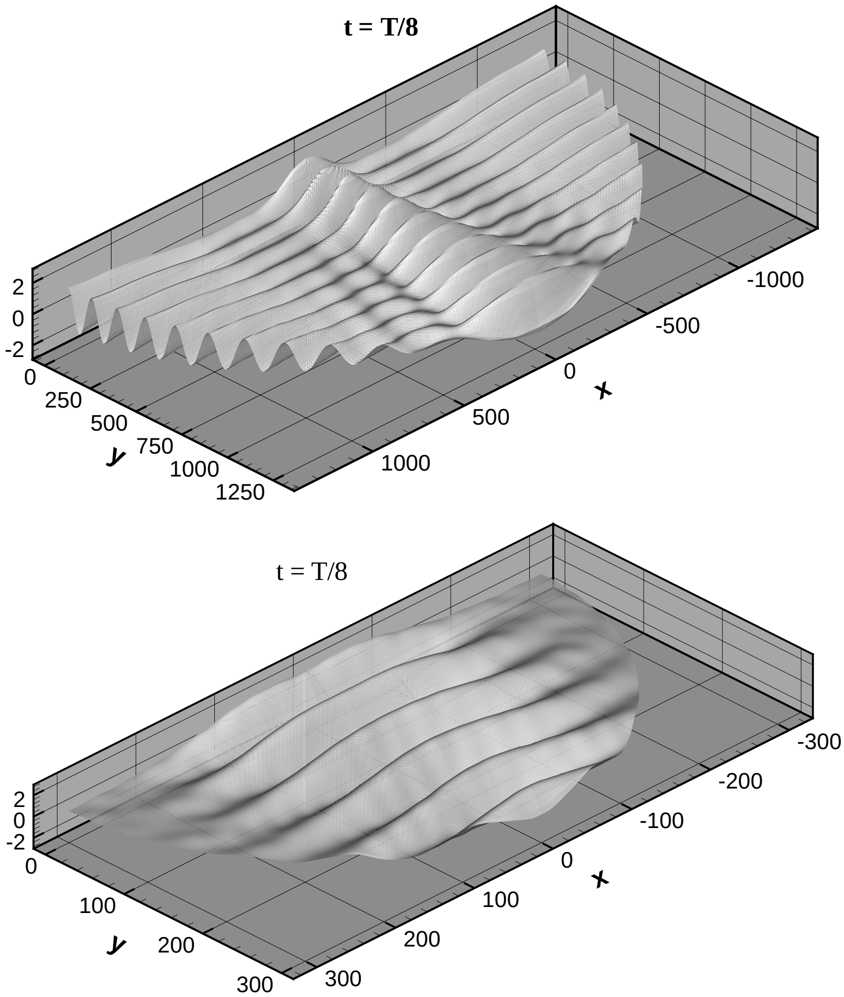

Figure 2: Sample surface contours at $t = T/8$ of (a) first order, first harmonic and (b)second order, second harmonic due to normally incident waves over a circular shoal. The input frequency is $\omega = 2\pi/T = 0.7$. Outer radius of shoal = 300 m. Depth outside the shoal = 40 m. Depth = 20 m at $r \leq 30$ m. The results are normalized by A and kA^2 with $k = 0.052$, separately. Figure (b) is magnified five times relative to Figure(a).

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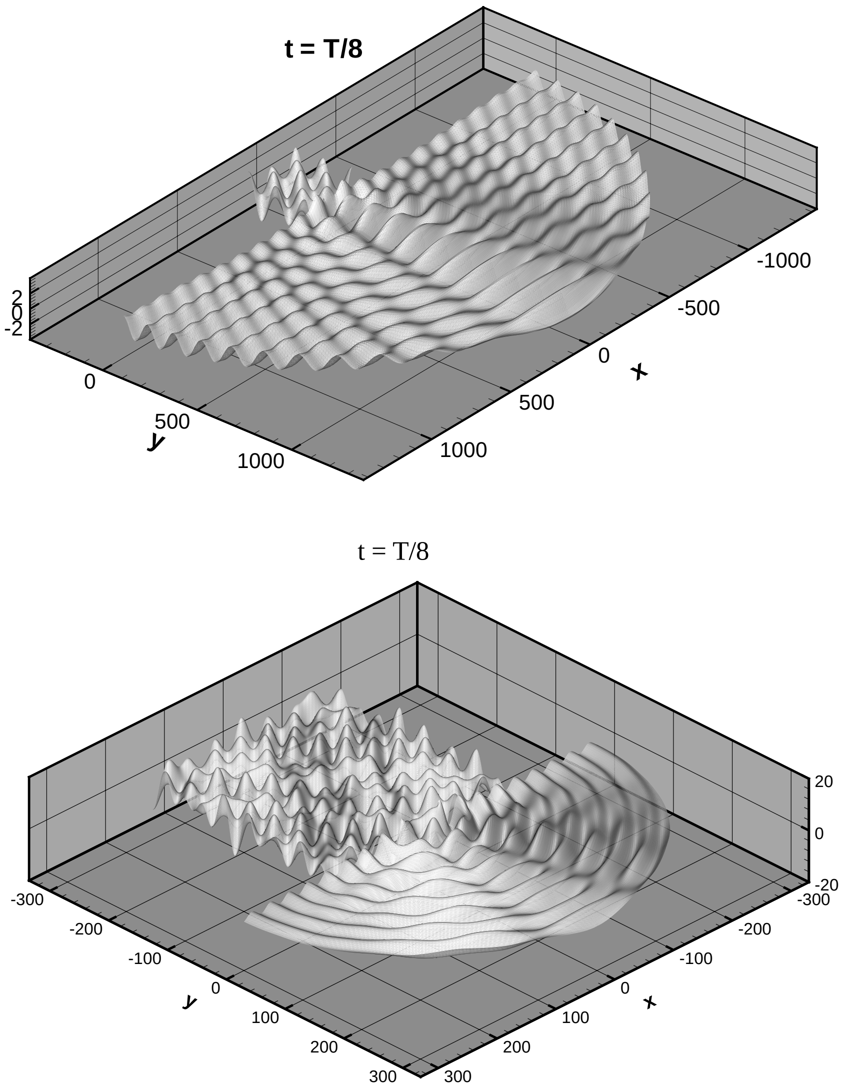

Figure 3: Sample surface contours at $t = T/8$ of (a) first order, first harmonic (b) second order, second harmonic due to normally incident waves with input frequency $\omega = 2\pi/T = 0.7$. There is a semi-circular shoal outside the harbor. Outer radius of shoal = 300 m. Depth outside the shoal = 40 m. Depth at the harbor entrance and inside the harbor = 20 m. The results are normalized by A and kA^2 with $k = 0.052$, separately. Second-order contours outside the semi-circular shoal,are not shown. Figure (b) is magnified five times relative to Figure(a).