A long-wave multiple scattering theory

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SUMMARY

One of the most successful techniques for handling multiple scattering is the so-called T-matrix approach in which the full linear solution can be computed once it has been determined how each individual element of an array scatters an arbitrary incident field. In the context of water waves this theory was originally formulated in [1] for constant finite depth, and extended to the deep water case in [2].

Here we propose a very simple approach to multiple scattering, designed to provide approximations valid when the wavelength is large compared to the size of the individual scatterers. The idea is an extension of a very old method used in acoustics due to Foldy [3] in which the scatterers are assumed to behave like point sources in the long-wave limit. This is in fact rigorously true for scatterers on whose boundary the velocity potential vanishes, but it is not appropriate for rigid scatterers. For our problem we assume that each scatterer can be modelled as a combination of a source and a dipole in long waves and then proceed to handle the multiple scattering in much the same way as in the T-matrix approach. The scattering characteristics of each individual scatterer can be determined by appealing to specific long-wave asymptotics, or numerically if necessary. A time dependence of $\exp(-\mathrm{i}\omega t)$ is assumed throughout and we will use $K = \omega^2/g$.

The method is applied to the scattering of a plane wave by a group of horizontal, submerged, circular cylinders and by an infinite periodic row of identical vertical cylinders of constant cross-section.

FORMULATION

We take the x, y-plane to be the undisturbed free surface with z pointing vertically upwards and represent the total field by the harmonic velocity potential

$$u(\mathbf{r}) = u_{\mathrm{inc}}(\mathbf{r}) + \sum_j \big\{ D_j G_0(\mathbf{r} - \mathbf{r}_j; z_j) + \mathbf{d}_j \cdot \mathbf{G}_1(\mathbf{r} - \mathbf{r}_j; z_j) \big\}, \quad (1)$$

where the sum is over all scatterers, the j-th scatterer being centred at $\mathbf{r}_j = (x_j, y_j, z_j)$, and $\mathbf{r} = (x, y, z)$. The first term inside the summation is a source at $\mathbf{r}_j$; the strength of the source (given by D_j) is unknown. The second term is a dipole at $\mathbf{r}_j$, the direction and strength of which (given by $\mathbf{d}_j = (d_j^x, d_j^y, d_j^z)$) are unknown. For submerged structures we do not include a source (i.e. $D_j = 0$).

The field incident on the n-th scatterer is

$$u_n(\mathbf{r}) = u_{\mathrm{inc}}(\mathbf{r}) + \sum_{j \neq n} \big\{ D_j G_0(\mathbf{r} - \mathbf{r}_j; z_j) + \mathbf{d}_j \cdot \mathbf{G}_1(\mathbf{r} - \mathbf{r}_j; z_j) \big\}. \quad (2)$$

Now, let us characterise the scattering properties of the scatterers by writing

$$D_n = B_n u_n(\mathbf{r}_n) \quad \text{and} \quad \mathbf{d}_n^T = \mathbf{C}_n \left[\mathbf{v}_n(\mathbf{r}_n) \right]^T, \quad (3)$$

where

$$\mathbf{v}_n = K^{-1} \nabla u_n. \quad (4)$$

The quantity $\mathbf{C}_n$ is a matrix. Thus D_n is proportional to the value of the exciting field at $\mathbf{r}_n$ and $\mathbf{d}_n$ is related to the gradient of the exciting field at $\mathbf{r}_n$.

If we substitute from (2) into (3) we get

$$D_n = B_n \Big[u_{\mathrm{inc}}(\mathbf{r}_n) + \sum_{j \neq n} \big\{ D_j G_0(\mathbf{r}_{nj}; z_j) + \mathbf{d}_j \cdot \mathbf{G}_1(\mathbf{r}_{nj}; z_j) \big\} \Big], \quad (5)$$

where $\mathbf{r}_{nj} = \mathbf{r}_n - \mathbf{r}_j$, and

$$\mathbf{d}_n^T = \mathbf{C}_n \Big[\mathbf{v}_{\mathrm{inc}}(\mathbf{r}_n) + \frac{1}{K} \sum_{j \neq n} \nabla \big(D_j G_0(\mathbf{r} - \mathbf{r}_j; z_j) + \mathbf{d}_j \cdot \mathbf{G}_1(\mathbf{r} - \mathbf{r}_j; z_j) \big)_{\mathbf{r} = \mathbf{r}_n} \Big]^T, \quad (6)$$

where $\mathbf{v}_{\mathrm{inc}}(\mathbf{r}) = K^{-1} \nabla u_{\mathrm{inc}}$. Equations (5) and (6) give a system of linear algebraic equations for D_n and the components of $\mathbf{d}_n$. For N scatterers in three dimensions, there are $4N$ equations for the $4N$ scalar unknowns; in two dimensions, there are $3N$ equations in $3N$ unknowns, though for each scatterer that is submerged the size of the system reduces by one.

Choice of B_j and $\mathbf{C}_j$

In order to use the method described above, we have to specify the coefficient B_j and the matrix $\mathbf{C}_j$ for each scatterer. Consider the j-th scatterer and assume, without loss of generality, that it is located at $\mathbf{r}_j = (0, 0, \zeta)$. For any incident field $u_{\mathrm{inc}}(\mathbf{r})$, we have assumed that the total field near the scatterer is given by

$$u(\mathbf{r}) \simeq u_{\mathrm{inc}}(\mathbf{r}) + B_j u_{\mathrm{inc}}(0, 0, \zeta)\, G_0(\mathbf{r}; z_j) + \left[\mathbf{v}_{\mathrm{inc}}(0, 0, \zeta)\, \mathbf{C}_j^T \right] \cdot \mathbf{G}_1(\mathbf{r}; z_j). \quad (7)$$

If the incident field is a plane wave travelling in the x-direction, then we have

$$u_{\text{inc}} = \mathrm{e}^{\mathrm{i}Kx}\mathrm{e}^{Kz}, \tag{8}$$

and

$$u(\mathbf{r}) \simeq \mathrm{e}^{\mathrm{i}Kx}\mathrm{e}^{Kz} + B_j \mathrm{e}^{K\zeta}\, G_0(\mathbf{r}; z_j) + \mathrm{e}^{K\zeta}\left[(\mathrm{i}, 0, 1)\mathbf{C}_j^T\right] \cdot \mathbf{G}_1(\mathbf{r}; z_j). \tag{9}$$

This can be compared with specific long-wave calculations such as those derived in [4] using matched asymptotic expansions.

SUBMERGED HORIZONTAL CYLINDERS

As an application of the theory we will consider a two-dimensional problem with a plane wave normally incident on an array of submerged horizontal circular cylinders in deep water. Since the cylinders are submerged, there are no source terms. Symmetry considerations show that the vertical fluid velocity cannot lead to a horizontal dipole, so $C_{zx} = 0$. We also assume that $C_{xz} = 0$; though this is an approximation which is strictly only valid when the depth of submergence of each cylinder, z_n, is much larger than the cylinder radius, a_n. It follows from the asymptotic analysis in [5], that for a circular cylinder of radius a and submergence depth $f > 0$,

$$C_{xx} = \delta(Ka)^2, \qquad C_{zz} = -\delta(Ka)^2, \tag{10}$$

where

$$\delta = 4\left((f/a)^2 - 1\right)\sum_{n=1}^{\infty}\frac{ns^{2n}}{1 - s^{2n}}, \tag{11}$$

with $s = (f/a) - \sqrt{(f/a)^2 - 1}$, is a factor which tends to one as $a/f \to 0$. For deeply submerged cylinders we have $\delta \approx 1$ and different cross-sections can easily be accommodated by including an appropriate dipole coefficient in (10) as shown in [4].

Equation (6) is thus

$$\begin{pmatrix} d_n^x \\ d_n^z \end{pmatrix} = \delta_n (Ka_n)^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left[(\mathrm{i}, 1)\, \mathrm{e}^{\mathrm{i}Kx_n}\mathrm{e}^{Kz_n} + \frac{1}{K}\sum_{j \neq n} \nabla\big(\mathbf{d}_j \cdot \mathbf{G}_1(\mathbf{r} - \mathbf{r}_j; z_j)\big)_{\mathbf{r}=\mathbf{r}_n}\right]^T. \tag{12}$$

Now

$$\mathbf{d}_j \cdot \mathbf{G}_1(\mathbf{r} - \mathbf{r}_j; z_j) = \frac{d_j^x(x - x_j) + d_j^z(z_j - z)}{Kr^2} + \fint_0^\infty \frac{(\mu + K)}{K(\mu - K)}\, \mathrm{e}^{\mu(z+z_j)} \left(d_j^x \sin\mu(x - x_j) - d_j^z \cos\mu(x - x_j)\right) \mathrm{d}\mu \tag{13}$$

and if we define

$$x_{nj} = x_n - x_j, \quad z_{nj} = z_n - z_j, \quad r_{nj} = \sqrt{x_{nj}^2 + z_{nj}^2}$$

and

$$\begin{pmatrix} I_{nj}^s \\ I_{nj}^a \end{pmatrix} = \fint_0^\infty \frac{\mu(\mu + K)}{K^2(\mu - K)}\, \mathrm{e}^{\mu(z_n + z_j)} \begin{pmatrix} \cos\mu x_{nj} \\ \sin\mu x_{nj} \end{pmatrix} \mathrm{d}\mu \tag{14}$$

then (12) becomes

$$\begin{pmatrix} d_n^x \\ d_n^z \end{pmatrix} = \delta_n (Ka_n)^2 \Bigg[\begin{pmatrix} \mathrm{i} \\ -1 \end{pmatrix} \mathrm{e}^{\mathrm{i}Kx_n}\mathrm{e}^{Kz_n} + \sum_{j \neq n} \Bigg[\frac{1}{K^2 r_{nj}^4} \begin{pmatrix} d_j^x(z_{nj}^2 - x_{nj}^2) + 2d_j^z x_{nj} z_{nj} \\ 2d_j^x x_{nj} z_{nj} + d_j^z(x_{nj}^2 - z_{nj}^2) \end{pmatrix} + \begin{pmatrix} d_j^x I_{nj}^s + d_j^z I_{nj}^a \\ -d_j^x I_{nj}^a + d_j^z I_{nj}^s \end{pmatrix} \Bigg]\Bigg]. \tag{15}$$

For N cylinders this is a $2N \times 2N$ system of equations for the unknown d_n^x and d_n^z. An approximate solution can be determined if we assume that both $Ka_n \ll 1$ and $a_n/r_{nj} \ll 1$. We obtain

$$\begin{pmatrix} d_n^x \\ d_n^z \end{pmatrix} \simeq \delta_n (Ka_n)^2 \Bigg[\begin{pmatrix} \mathrm{i} \\ -1 \end{pmatrix} \mathrm{e}^{\mathrm{i}Kx_n}\mathrm{e}^{Kz_n} + \sum_{j \neq n} \delta_j (Ka_j)^2 \mathrm{e}^{\mathrm{i}Kx_j}\mathrm{e}^{Kz_j} \Bigg[\frac{1}{K^2 r_{nj}^4} \begin{pmatrix} \mathrm{i}(z_{nj}^2 - x_{nj}^2) - 2x_{nj}z_{nj} \\ 2\mathrm{i}x_{nj}z_{nj} - (x_{nj}^2 - z_{nj}^2) \end{pmatrix} + \begin{pmatrix} \mathrm{i}I_{nj}^s - I_{nj}^a \\ -\mathrm{i}I_{nj}^a - I_{nj}^s \end{pmatrix} \Bigg]\Bigg]. \tag{16}$$

The scattered field is

$$u_{\text{sc}} = \sum_n \mathbf{d}_j \cdot \mathbf{G}_1(\mathbf{r} - \mathbf{r}_n; z_n) \tag{17}$$

$$\sim 2\pi \sum_n \mathrm{e}^{\pm\mathrm{i}K(x - x_n)}\mathrm{e}^{K(z + z_n)}(\pm d_n^x - \mathrm{i}d_n^z) \tag{18}$$

as $x \to \pm\infty$. The reflection coefficient is thus

$$\mathcal{R} = 2\pi \sum_n \mathrm{e}^{\mathrm{i}Kx_n}\mathrm{e}^{Kz_n}(-d_n^x - \mathrm{i}d_n^z) \tag{19}$$

which becomes

$$\mathcal{R} = -4\pi \sum_n \sum_{j \neq n} \mathrm{e}^{\mathrm{i}K(x_n + x_j)}\mathrm{e}^{K(z_n + z_j)} \times \frac{\delta_n \delta_j (Ka_j a_n)^2}{r_{nj}^4}\left(\mathrm{i}(z_{nj}^2 - x_{nj}^2) - 2x_{nj}z_{nj}\right) \tag{20}$$

if we insert the approximations given by (16). If all the cylinders are at the same depth $z_n = \zeta$ and have the same radius $a_n = a$, then

$$\mathcal{R} = 4\pi\mathrm{i}K^2 a^4 \delta^2 \mathrm{e}^{2K\zeta} \sum_n \sum_{j \neq n} x_{nj}^{-2}\, \mathrm{e}^{\mathrm{i}K(x_n + x_j)}. \tag{21}$$

Furthermore, if the wavelength is large compared with all the other length scales in the problem we get

$$\mathcal{R} = 4\pi \mathrm{i} K^2 a^4 \delta^2 \sum_n \sum_{j\neq n} x_{nj}^{-2}. \tag{22}$$

This is slightly different to the equivalent expression in [6], where the factor δ^2 is replaced by $\delta\tilde{\delta}$, $\tilde{\delta}$ being a different factor which also tends to 1 as $a/\zeta \to 0$. This discrepancy, which makes little difference to the numerical results, may well be due to the neglect of the C_{xz} terms in the matrix $\mathbf{C}$.

Some preliminary calculations of exciting forces have been performed based on solving (15) and then numerically integrating the potential (1) around each cylinder. These results have been compared with the full linear solution computed using the multipole expansion method described in [7]. As expected, the two approaches agree in the long wave limit.

VERTICAL CYLINDERS

In the case of vertical cylinders of constant cross section extending throughout the depth (h say) we can develop a very similar theory. A depth dependence of $\cosh k(z+h)/\cosh kh$, where k the positive root of the dispersion relation $k \tanh kh = K$, can be factored out in the usual way and then the reduced potential (which we still call u) satisfies the Helmholtz equation $(\nabla^2 + k^2)u = 0$. The definition of $\mathbf{v}_n$ is changed to $k^{-1}\nabla u_n$. Infinite depth is treated by setting $k = K$ with the depth factor as $\exp(-Kz)$.

Equation (1) remains the same, but now $G_0(\mathbf{r}) = H_0^{(1)}(kr)$ and $\mathbf{G}_1(\mathbf{r}) = \hat{\mathbf{r}} H_1^{(1)}(kr)$, with $r = |\mathbf{r}|$ and $\hat{\mathbf{r}} = \mathbf{r}/r$. Equations (5) and (6) become (dropping the superscripts on the Hankel functions)

$$D_n = B_n \Big[u_{\mathrm{inc}}(\mathbf{r}_n) + \sum_{j\neq n} \big\{ D_j H_0(kr_{nj}) + \mathbf{d}_j \cdot \hat{\mathbf{r}}_{nj}\, H_1(kr_{nj}) \big\} \Big] \tag{23}$$

and

$$\mathbf{d}_n^T = \mathbf{C}_n \Big[\mathbf{v}_{\mathrm{inc}}(\mathbf{r}_n) + \sum_{j\neq n} \Big\{ \frac{H_1(kr_{nj})}{kr_{nj}}\, \mathbf{d}_j - \hat{\mathbf{r}}_{nj}\, (\mathbf{d}_j \cdot \hat{\mathbf{r}}_{nj})\, H_2(kr_{nj}) - D_j \hat{\mathbf{r}}_{nj}\, H_1(kr_{nj}) \Big\} \Big]^T. \tag{24}$$

Standard low frequency approximations show that for circular cylinders

$$B_j = -\tfrac{1}{4}\mathrm{i}\pi (ka_j)^2, \qquad \mathbf{C}_j = -2B_j \mathbf{I}, \tag{25}$$

where $\mathbf{I}$ is the identity matrix. Equivalent quantities can easily be determined for cylinders of arbitrary cross-section. For cylinders with a cross-section that is symmetric with respect to both the x- and y-axes the matrix $\mathbf{C}$ will be diagonal and we will make this assumption here.

As an example, we consider the scattering of a plane wave

$$u_{\mathrm{inc}} = \mathrm{e}^{\mathrm{i}(\beta x + \alpha y)}, \tag{26}$$

where $\alpha = k \sin\psi$ and $\beta = k\cos\psi$, by an infinite periodic row of identical cylinders. The scatterers are located at $\mathbf{r} = \mathbf{r}_m$ for $m = 0, \pm1, \pm2, \ldots$, where $\mathbf{r} = (x, y)$, $\mathbf{r}_m = (ms, 0)$ and s is the spacing. We will use polar coordinates (r_m, θ_m) centred at the m-th scatterer and defined by $x - ms = r_m \cos\theta_m$, $y = r_m \sin\theta_m$. In terms of (r_m, θ_m), we have

$$u_{\mathrm{inc}} = I_m \mathrm{e}^{\mathrm{i}kr_m \cos(\theta_m - \psi)} \quad \text{with} \quad I_m = \mathrm{e}^{\mathrm{i}\beta ms}. \tag{27}$$

The representation (1) becomes

$$u(\mathbf{r}) = u_{\mathrm{inc}}(\mathbf{r}) + \sum_j \big\{ D_j H_0(kr_j) + (d_j^x \cos\theta_j + d_j^y \sin\theta_j) H_1(kr_j) \big\}. \tag{28}$$

As $u_{\mathrm{inc}}(ns, 0) = I_n$, the scalar system (23) becomes

$$D_n = B \Big[I_n + \sum_{j\neq n} \big\{ D_j H_0(ks|n-j|) + d_j^x H_1(ks|n-j|)\,\mathrm{sgn}(n-j) \big\} \Big]. \tag{29}$$

Note that $\hat{\mathbf{r}}_{nj} = (1, 0)$ for $n > j$ and $\hat{\mathbf{r}}_{nj} = (-1, 0)$ for $n < j$. The vector system (24) reduces to two scalar systems. They are

$$C_{xx} d_n^x = \mathrm{i} I_n \cos\psi + \sum_{j\neq n} \big\{ d_j^x H_1'(ks|n-j|) - D_j H_1(ks|n-j|)\,\mathrm{sgn}(n-j) \big\} \tag{30}$$

and

$$C_{yy} d_n^y = \mathrm{i} I_n \sin\psi + \sum_{j\neq n} d_j^y \frac{H_1(ks|n-j|)}{ks|n-j|}. \tag{31}$$

The periodicity of the geometry and the quasi-periodicity of the incident plane wave imply

$$D_j = I_j D_0, \quad d_j^x = I_j d_0^x, \quad d_j^y = I_j d_0^y. \tag{32}$$

When these relations are used in (29), (30)and (31) we obtain a coupled system for D_0 and d_0^x, representing the component of the solution which is symmetric with respect to the x-axis, and a separate equation for the antisymmetric component d_0^y. The solutions are

$$\begin{aligned} D_0 &= \big\{ C_{xx} - \tfrac{1}{2}(\sigma_0 - \sigma_2) + \mathrm{i}\sigma_1 \cos\psi \big\} / \Delta, \\ d_0^x &= \big\{ -\sigma_1 + (B^{-1} - \sigma_0)\mathrm{i}\cos\psi \big\} / \Delta, \\ d_0^y &= \mathrm{i}\sin\psi \big\{ C_{yy} - \tfrac{1}{2}(\sigma_0 + \sigma_2) \big\}^{-1}, \end{aligned} \tag{33}$$

where $\Delta = (B^{-1} - \sigma_0)[C_{xx} - \frac{1}{2}(\sigma_0 - \sigma_2)] + \sigma_1^2$, and

$$\sigma_p(\psi) = \sum_{j=1}^{\infty} (I_{-j} + (-1)^p I_j) H_p(kjs). \tag{34}$$

The efficient computation of these sums is non-trivial, but integral representations for σ_p exist which greatly facilitate the process. If we define

$$S_n^{\pm} = \sum_{j=1}^{\infty} I_{\pm j} H_n(kjs) \tag{35}$$

so that $\sigma_n = (-1)^n S_n^+ + S_n^-$, then

$$S_n^{\pm} = -\frac{\mathrm{i}}{\pi} \int_C \frac{\mathrm{e}^{-\mathrm{i}n \arccos t}}{\gamma(\mathrm{e}^{ks\gamma \mp \mathrm{i}\beta s} - 1)} \, \mathrm{d}t, \tag{36}$$

where the contour C lies on the real axis but is indented above the poles for which $t < 0$ and below those for which $t > 0$. Here $\gamma(t)$ is defined for real t by

$$\gamma(t) = \begin{cases} -\mathrm{i}\sqrt{1-t^2}, & |t| < 1, \\ \sqrt{t^2 - 1}, & |t| > 1. \end{cases} \tag{37}$$

and for $t \in \mathbb{C}$ we have branch cuts from 1 to $1 + \mathrm{i}\infty$ and from -1 to $-1 - \mathrm{i}\infty$.

From (28) and (32), we obtain the representation

$$u(\mathbf{r}) = u_{\text{inc}}(\mathbf{r}) + \sum_j I_j \big\{ D_0 H_0(kr_j) + (d_0^x \cos\theta_j + d_0^y \sin\theta_j) H_1(kr_j) \big\}. \tag{38}$$

We shall evaluate this expression in the far field in order to determine the reflection and transmission coefficients for the problem. First, we define the scattering angles

$$\psi_m = \arccos(\beta_m / k) \quad \text{with} \quad \beta_m = \beta + 2m\pi/s. \tag{39}$$

If $|\beta_m| < k$, we write $m \in \mathcal{M}$ and then $0 < \psi_m < \pi$. Using integral representations for the Hankel functions in (38) and then applying the Poisson summation formula, we find that

$$u = u_{\text{inc}} + 2 \sum_m \frac{\mathrm{e}^{\mathrm{i}kr \cos(\theta - \mathrm{sgn}(y)\,\psi_m)}}{ks \sin\psi_m} \times \big\{ D_0 - \mathrm{i}d_0^x \cos\psi_m - \mathrm{i}\,\mathrm{sgn}(y)\, d_0^y \sin\psi_m \big\}. \tag{40}$$

For those m for which $|\beta_m/k| > 1$, we have $\mathrm{i}k \sin\psi_m = -\sqrt{\beta_m^2 - k^2}$. Thus, the terms in the sum for these values of m decay rapidly as $|y| \to \infty$. Hence, the far field involves only those m for which $m \in \mathcal{M}$. As we are interested in long waves we can assume that $ks < \pi$ in which case $\mathcal{M} = \{0\}$ and we have just one reflected and one transmitted wave, with reflection and transmission coefficients given by

$$\mathcal{R} = \frac{2}{ks \sin\psi} \big\{ D_0 - \mathrm{i}d_0^x \cos\psi + \mathrm{i}d_0^y \sin\psi \big\}, \tag{41}$$

$$\mathcal{T} = 1 + \frac{2}{ks \sin\psi} \big\{ D_0 - \mathrm{i}d_0^x \cos\psi - \mathrm{i}d_0^y \sin\psi \big\}. \tag{42}$$

DISCUSSION

Calculations based on this long-wave multiple scattering theory will be presented at the workshop for the two problems considered above. Comparisons will be made with the full linear solution in each case. We will also discuss the three-dimensional water wave problem involving an array of floating hemispheres.

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