The two-dimensional water-wave problem for multiple finite docks

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1 Introduction; statement of the problem

Problems concerning the interaction of water waves with a *single* rigid plate of finite extent and infinite length lying in the free surface have a long history described by Linton in [1], where some reasons, why problems for finite docks are interesting, are also discussed. Much less is known about the water-wave problem for *multiple* finite docks, even in the simplest, two-dimensional case. So far, the only publication found by the author on this topic is the old paper [2], in which Haskind proposed a method providing an approximate two-dimensional solution for deep water. The aim of the present note is to fill in the gap (at least in the part dealing with multiple docks), that exists in mathematical studies of the interaction of water waves with multiple obstacles floating in the free surface.

Let the cross-section of a domain occupied by an inviscid, incompressible, heavy fluid (water) be

$$\mathbb{R}^2_+ = \{-\infty < x < +\infty, y > 0\},$$

and so the undisturbed free surface lies in the plane $y = 0$, whereas the y-axis directed vertically upwards. For defining $M \geq 2$ rigid, finite docks floating in the water surface we take $2M$ points on the x-axis:

$$a_1 < b_1 < a_2 < b_2 < \ldots < a_M < b_M, \quad a_1 \neq -\infty, b_M \neq +\infty,$$

as the endpoints of the docks. Then two sets $D = \{x \in (a_1, b_1) \cup (a_2, b_2) \cup \ldots \cup (a_M, b_M);\ y = 0\}$ and $F = \partial\mathbb{R}^2_+ \setminus \overline{D}$ correspond to the union of M docks and to the free surface, respectively.

If the surface tension is neglected and the water motion is assumed to be irrotational and two-dimensional (the latter means that the motion is the same in every plane orthogonal to the (x, y)-plane), then the small-amplitude oscillations of water such that the radian frequency is equal to ω are described mathematically by a complex-valued velocity potential ϕ satisfying the following boundary value problem:

$$\nabla^2\phi = 0 \ \text{ in } \mathbb{R}^2_+, \quad \phi_y + \nu\phi = 0 \ \text{ on } F, \quad \phi_y = f \ \text{ on } D, \tag{1}$$

$$\phi_x \mp i\nu\phi = o(1) \ \text{ uniformly in } y \in [0, +\infty) \text{ as } \pm x \to +\infty. \tag{2}$$

This problem must be complemented by the condition that the Dirichlet integral of ϕ is locally finite. In the last condition (1), f is a given function depending on the type of problem (radiation or scattering); ν is a nonnegative spectral parameter equal to ω^2/g, where g is the acceleration due to gravity acting along the y-axis.

2 The uniqueness theorem

Since we are interested in the question of uniqueness, we assume that $f = 0$ in (1) throughout this section, and so ϕ denotes a solution to the homogeneous boundary value problem. It is clear that the difference of two solutions satisfies the homogeneous Neumann condition on D because problem (1), (2) is linear.

It is known (see *e.g.*, [3], Section 2.2) that a solution to the homogeneous problem satisfies the following conditions:

$$\phi(x, y) = O\left([x^2 + y^2]^{-1/2}\right) \quad \text{and} \quad |\nabla\phi| = O\left([x^2 + y^2]^{-1}\right) \quad \text{as } x^2 + y^2 \to \infty, \tag{3}$$

which are more restrictive than (2). Furthermore, the fact that the Dirichlet integral of ϕ is locally finite allows us to apply the standard technique (see *e.g.*, [4], Chapter 2) for obtaining the asymptotic formulae near the tips of docks. The following formula is valid near $(a_m, 0)$, $m = 1, \ldots, M$:

$$\phi(\rho, \theta) = c_m \left[\pi\nu^{-1} + \rho(\cos\theta \, \log\rho - \theta\sin\theta)\right] + d_m \rho\cos\theta + \psi_m(\rho, \theta). \tag{4}$$

Here (ρ, θ) are polar coordinates such that $x + iy = a_m + \rho e^{i\theta}$, c_m and d_m are constants, and the estimates $\psi_m = O\left(\rho^{1+\delta}\right)$, $\nabla\psi_m = O\left(\rho^{\delta}\right)$ hold as $\rho \to 0$, where $\delta > 0$. Formulae (3), (4), and the asymptotic formulae similar to (4) valid near $(b_m, 0)$, $m = 1, \ldots, M$, yield

$$\int_{\boldsymbol{R}^2_+} |\nabla\phi|^2 \, \mathrm{d}x\mathrm{d}y < \infty, \qquad \int_F \phi^2 \, \mathrm{d}x < \infty. \tag{5}$$

Here the first (second) integral is proportional to the kinetic (potential) energy of waves per unit length of the docks' generators. Moreover, conditions (5) can be imposed instead of (2) for the homogeneous boundary value problem, and so ϕ can be assumed to be real because if it were complex, then both the real and imaginary parts would separately satisfy the problem.

Proposition 1. *Let $M = 2$ and $b_1 = -a_2$, $a_1 = -b_2$, i.e., there are two docks symmetric about the y-axis. Then the homogeneous boundary value problem* (1), (5) *has only a trivial antisymmetric* ($\phi(-x, y) = -\phi(x, y)$) *solution.*

A sketch of proof. Our proof of this theorem is based on two ideas. First we use an appropriate conformal mapping and then apply the so-called Vainberg–Maz'ya identity (see [3], Section 2.2.2).

For antisymmetric solutions, it is sufficient to consider ϕ only in $Q = \{x > 0, y > 0\}$ and use the boundary condition $\phi = 0$ on $S = \{x = 0, y > 0\}$. Without loss of generality, we put $a_2 = 1$ and write b instead of b_2. It is well-known that $w = \sqrt{z^2 - 1}$ ($z = x + iy$, $w = u + iv$) maps Q onto Q so that $\mathcal{S} = \{u = 0, v > 1\}$, $\mathcal{F}^{(+)}_0 = \{u = 0, 0 < v < 1\}$, $\mathcal{F}^{(+)}_\infty = \{u > \sqrt{b^2 - 1}, v = 0\}$, and $\mathcal{D}^{(+)} = \{0 < u < \sqrt{b^2 - 1}, v = 0\}$ are the images of S, $F^{(+)}_0 = \{0 < x < 1, y = 0\}$, $F^{(+)}_\infty = \{x > b, y = 0\}$, and $D^{(+)} = \{1 < x < b, y = 0\}$, respectively.

Let $\varphi(u, v) = \phi(x(u, v), y(u, v))$. It is clear that φ is harmonic in Q and satisfies the homogeneous Dirichlet and Neumann conditions on $\mathcal{S}$ and $\mathcal{D}^{(+)}$, respectively. Moreover, we have:

$$\varphi_v + \nu \frac{u}{\sqrt{1 + u^2}} \varphi = 0 \ \text{ on } \ \mathcal{F}^{(+)}_\infty \quad \text{and} \quad \varphi_u + \nu \frac{v}{\sqrt{1 - v^2}} \varphi = 0 \ \text{ on } \ \mathcal{F}^{(+)}_0.$$

Therefore, the equation $\int_Q |\nabla\phi|^2 \, \mathrm{d}x\mathrm{d}y = \nu \int_{F^{(+)}_0 \cup F^{(+)}_\infty} \phi^2 \, \mathrm{d}x$, expressing the equipartition of energy for the homogeneous problem, takes the following form for φ:

$$\int_Q |\nabla\varphi|^2 \, \mathrm{d}u\mathrm{d}v = \nu \left[\int_0^1 \varphi^2(0, v) \frac{v \, \mathrm{d}v}{\sqrt{1 - v^2}} + \int_{\sqrt{b^2 - 1}}^{+\infty} \varphi^2(u, 0) \frac{u \, \mathrm{d}u}{\sqrt{1 + u^2}}\right]. \tag{6}$$

We will use two versions of the Vainberg–Maz'ya identity:

$$(2v\varphi_v + \varphi) \, \nabla^2\varphi = \nabla \cdot (2v\varphi_v + \varphi) \, \nabla\varphi - 2\varphi_v^2 - \left(v|\nabla\varphi|^2\right)_v,$$

and another one, where v is changed to u. Since φ is harmonic, integrating the above identity over Q and applying the divergence theorem, we get

$$2\int_Q \varphi_v^2 \, \mathrm{d}u\mathrm{d}v = \int_{\partial Q} (2v\varphi_v + \varphi) \, \partial_n\varphi \, \mathrm{d}s = \nu \left[\int_{F^{(+)}_\infty} \varphi^2 \, \frac{u \, \mathrm{d}u}{\sqrt{1 + u^2}} + \int_{\mathcal{F}^{(+)}_0} (2v\varphi_v + \varphi) \, \frac{\varphi v \, \mathrm{d}v}{\sqrt{1 - v^2}}\right].$$

Here ∂_n indicates differentiation with respect to the exterior normal and the second equality follows when the boundary conditions are used. Integration by parts yields

$$\int_{\mathcal{F}^{(+)}_0} 2v\varphi_v \, \frac{\varphi v \, \mathrm{d}v}{\sqrt{1 - v^2}} = -\int_{\mathcal{F}^{(+)}_0} \varphi^2 \, \frac{v \, \mathrm{d}v}{\sqrt{1 - v^2}} - \int_{\mathcal{F}^{(+)}_0} \varphi^2 \, \frac{v \, \mathrm{d}v}{(1 - v^2)^{3/2}},$$

since the integrated terms vanish. This is obvious at $v = 0$ and at $v = 1$ this is a consequence of the estimate $\varphi(0, v) = O\left(\sqrt{1 - v^2}\right)$ as $v \to 1 - 0$, holding in view of $\phi(z) = O(|z|)$ as $|z| \to 0$ (the latter estimate is true because ϕ is antisymmetric). Thus we arrive at the following identity:

$$2\int_Q \varphi_v^2 \, \mathrm{d}u\mathrm{d}v + \nu \int_{\mathcal{F}^{(+)}_0} \varphi^2 \, \frac{v \, \mathrm{d}v}{(1 - v^2)^{3/2}} = \nu \int_{F^{(+)}_\infty} \varphi^2 \, \frac{u \, \mathrm{d}u}{\sqrt{1 + u^2}}.$$

The second Vainberg–Maz'ya identity leads in the same way to the identity

$$2\int_Q \varphi_u^2 \, \mathrm{d}u\mathrm{d}v + \nu \left[\frac{b^2-1}{b} \, \varphi^2 \left(\sqrt{b^2-1}, 0 \right) + \int_{\mathcal{F}_\infty^{(+)}} \varphi^2 \, \frac{u \, \mathrm{d}u}{(1+u^2)^{3/2}} \right] = \nu \int_{F_0^{(+)}} \varphi^2 \, \frac{v \, \mathrm{d}v}{\sqrt{1-v^2}} .$$

Subtracting (6) from the sum of the last two identities, we obtain

$$\int_Q |\nabla\varphi|^2 \, \mathrm{d}u\mathrm{d}v + \nu \left[\int_{\mathcal{F}_0^{(+)}} \varphi^2 \, \frac{v \, \mathrm{d}v}{(1-v^2)^{3/2}} + \frac{b^2-1}{b} \, \varphi^2 \left(\sqrt{b^2-1}, 0 \right) + \int_{\mathcal{F}_\infty^{(+)}} \varphi^2 \, \frac{u \, \mathrm{d}u}{(1+u^2)^{3/2}} \right] = 0,$$

which is impossible unless φ vanishes identically in $\overline{Q}$.

3 The solvability theorem via integral equations

Using the two-dimensional Green's function $G(z, \zeta)$, $z = x + iy$, $\zeta = \xi + i\eta \in \overline{I\!R^2_+}$, for water of infinite depth (see expressions in [3], Section 1.2.1, but some signs must be changed because the water domain is $I\!R^2_-$ there), one can apply various techniques involving integral equations to the water-wave problem for multiple finite docks. Since the water domain coincides with the whole $I\!R^2_+$, integral equations arising for this problem are free of irregular frequencies provided the uniqueness theorem holds. (See Sections 3.1.1 and 3.1.2 in [3], where the notion of irregular frequencies is analysed.)

First we note that a solution to problem (1), (2) admits the representation:

$$-2\pi\phi(z) = \int_D \left[f(\xi) \, G(z,\xi) - \phi(\xi) \, G_\eta(z,\xi) \right] \mathrm{d}\xi \, , \quad z \in I\!R^2_+ . \tag{7}$$

Then, applying the jump formula for the double-layer potential as z tends to its limit position on D, one arrives at the integral equation:

$$-\phi(x) + \frac{1}{\pi} \int_D \phi(\xi) \, G_\eta(x,\xi) \, \mathrm{d}\xi = \frac{1}{\pi} \int_D f(\xi) \, G(x,\xi) \, \mathrm{d}\xi \, , \quad x \in D. \tag{8}$$

On the other hand, if one seeks a solution in the form:

$$\phi(z) = \int_D \mu(\xi) \, G(z,\xi) \, \mathrm{d}\xi \, , \quad z \in I\!R^2_+ , \tag{9}$$

then, applying the jump formula for the normal derivative of the single-layer potential as z tends to its limit position on D, one arrives at the integral equation:

$$-\mu(x) + \frac{1}{\pi} \int_D \mu(\xi) \, G_y(x,\xi) \, \mathrm{d}\xi = \frac{1}{\pi} f(x) \, , \quad x \in D. \tag{10}$$

Thus we have two integral equations for problem (1), (2).

Now we make an important observation that follows from formulae (1.48) and (1.49) in [3]:

$$G(x,\xi) = \mathcal{K}(x-\xi) \quad \text{and} \quad G_y(x,\xi) = G_\eta(x,\xi) = -\nu\mathcal{K}(x-\xi), \quad \text{where} \quad \mathcal{K}(x-\xi) = 2\int_{\ell_-} \frac{\cos k(x-\xi)}{k-\nu} \, \mathrm{d}k.$$

Here $x, \xi \in \partial I\!R^2_+$, the path of integration ℓ_- in the complex k-plane goes along the positive real axis and is indented below the pole $k = \nu$ (see [3], Section 1.1.1, Fig. 1.1). Hence the same integral operator K with the kernel $\mathcal{K}(x-\xi)$ acts on the unknowns ϕ and μ in the integral equations (8) and (10), respectively. In (8), K is also applied to f on the right-hand side, and it is convenient to have only one operator when solving this equation numerically.

Formula 3.722.7 in [5] allows us to express the kernel $\mathcal{K}(x-\xi)$ in terms of $\mathrm{Si}(X)$ and $\mathrm{Ci}(X)$, that are the sine and cosine integral, respectively:

$$\mathcal{K}(x-\xi) = -2 \left\{ \cos\nu(x-\xi) \, \mathrm{Ci}(\nu|x-\xi|) + \sin\nu|x-\xi| \left[(\pi/2) + \mathrm{Si}(\nu|x-\xi|) \right] \right\} + \pi i \cos\nu(x-\xi).$$

Since $\mathrm{Ci}(X)$ has logarithmic singularity at $X = 0$ (see [6], Chapter 5), the same is true for $\mathcal{K}(x - \xi)$ at $x = \xi$. Therefore, K is a compact operator in $C(\overline{D})$ and the Riesz–Fredholm theory (see *e.g.*, [7], Chapter 1) is applicable to equations (8) and (10).

Let us show that *if* μ_0 *satisfies the homogeneous equation* $\mu_0 = (\nu/\pi)\, K\mu_0$*, then* μ_0 *vanishes identically on* D *provided the uniqueness theorem holds for problem* (1), (2). Indeed, we have that ϕ_0 defined by formula (9) with the density μ_0 satisfies the homogeneous problem (1), (2) and, according to the uniqueness theorem, ϕ_0 vanishes identically in $\overline{\mathbb{R}^2_+}$. Then the continuity of the single-layer potential implies that $K\mu_0 = 0$, and so $\mu_0 = 0$ on D. Now the following result is a consequence of the Riesz–Fredholm theory.

Proposition 2. *Let the uniqueness theorem holds for problem* (1), (2). *Then the integral equations* (8) *and* (10) *have unique solutions for all right-hand-side terms belonging to* $C(\overline{D})$. *Using these solutions in formulae* (7) *and* (9)*, respectively, one obtains two representations of a unique solution to problem* (1), (2).

4 Discussion

Uniqueness for all values of the radian frequency has been established in the antisymmetric, two-dimensional water-wave problem involving a pair of docks of finite extent. Unfortunately, our method fails to prove the uniqueness theorem for the symmetric problem. It is worth to mention that the uniqueness theorem for all values of the radian frequency was earlier proved only for two geometries having a portion of the free surface separated from infinity (see [3], Section 4.2.1): 1) a strip of constant depth containing a pair of vertical, surface-piercing barriers (for this geometry uniqueness is proved for the oblique-wave problem); 2) a layer of constant depth containing a vertical, surface-piercing shell whose directrix is a closed piecewise smooth curve.

Along with uniqueness, we have proposed two Fredholm integral equations for finding a solution to the boundary value problem. The characteristic property of these equations is the fact that both of them contain the same integral operator, whose kernel is expressed explicitly in terms of the sine and cosine integrals.

In conclusion, let us turn to other techniques for the multiple-dock problem. Among them are Babinet's principle and the modified residue calculus. Much use of Babinet's principle is made in acoustic and electromagnetic theory, but MacCamy [8] proposed the extended version of this principle that has a wider range of applications. In particular, the water-wave problem for a single dock of finite extent is treated in [8] by means of this principle. Of course, MacCamy's method must be further modified in order to be applicable to problem (1), (2). The same is true for the modified residue calculus described in [1]. However, it looks plausible that the integral equation technique described here is preferable not only from the theoretical point of view, but also as a tool for developing numerical algorithms.

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