A variational model for fully non-linear water waves of Boussinesq type

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Abstract.
Using a variational principle and a parabolic approximation to the vertical structure of the velocity potential, the equations of motion for surface gravity waves over mildly sloping bathymetry are derived. No approximations are made concerning the non-linearity of the waves. The resulting model equations conserve mass, momentum and positive-definite energy. They are shown to have improved frequency-dispersion characteristics, as compared to classical Boussinesq-type of wave equations.

1. Introduction. Classical Boussinesq-type of models suffer from the introduction of high-order mixed spatial and temporal derivatives. Further, many Boussinesq-type models are not derived from variational principles and do not satisfy energy conservation. We want to obtain a model for non-linear waves which does not have these drawbacks.

High-order non-linear models, like Dommermuth & Yue (1987) and Agnon *et al.* (1999), solve free-surface evolution equations derived from a Hamiltonian under the constraint that the Laplace equation is satisfied exactly in the interior of the fluid domain. However, these models have to relate free-surface quantities to those at some fixed level. This is done by using truncated Taylor-series expansions, thus destroying the exact solution of the Laplace equation in the interior of the domain and with that the conservation of energy. Conservation of energy is important, since it can be related to high wave-number instabilities of the model.

Dingemans (1997) describes several methods for constructing Boussinesq-type models with positive-definite Hamiltonian, but these methods are quite tedious and have certain ambiguities regarding the order of certain operators, see also Broer (1974, 1975) and Broer *et al.* (1976). The described models are weakly non-linear.

The present method is easier and unambiguous, leads to a positive-definite Hamiltonian and can be fully non-linear if desired. Besides the fully non-linear form we also give a simpler weakly non-linear form. The drawback of the present model is, that instead of higher-order spatial and/or mixed spatial-temporal derivatives, an additional elliptic equation in the horizontal plane has to be solved (which is also the case for Agnon *et al.* 1999).

In the following, a parabolic approximation is used for the vertical distribution of horizontal velocity, since this is simple and eases the derivations. This parabolic approximation already gives improved linear dispersion characteristics as compared to classical Boussinesq-type models.

However, better performance can be achieved by choosing a power-series expansion in the vertical direction of the velocity potential. For each additional term in the power series an additional elliptic equation has to be solved, without increasing the order of the spatial derivatives in the model equations. Additional terms result in further improvement of the frequency-dispersion characteristics as well as non-linear behaviour of the model. However, the description of this model will be postponed for the moment.

2. Fully non-linear model. We start from the variational principle for irrotational water waves in the form as given by Miles (1977) (see also Milder 1977):

$$0 \;=\; \delta \mathcal{L} \;=\; \delta \iint L \, \mathrm{d}x \, \mathrm{d}t, \tag{2.1}$$

with $L(\zeta, \partial_t \zeta, \varphi, \partial_x \phi, \partial_z \phi; x, t)$ the Lagrangian density:

$$L \;=\; \varphi \, \partial_t \zeta \;-\; H \quad \text{with} \quad \varphi = [\phi]_{z=\zeta} \;, \tag{2.2}$$

where $\zeta(x,t)$ is the surface elevation, $\phi(x,z,t)$ is the velocity potential and the energy density $H(\zeta, \partial_x \phi, \partial_z \phi; x, t)$ is given by the sum of kinetic and potential energy densities:

$$H \;=\; \int_{-h}^{\zeta} \frac{1}{2} \left[(\partial_x \phi)^2 \;+\; (\partial_z \phi)^2 \right] \mathrm{d}z \;+\; \frac{1}{2}\, g\, \zeta^2, \tag{2.3}$$

while the mass density ρ is taken to be constant and equal to one. Further $h(x)$ is the still-water depth and g is the gravitational acceleration. This Lagrangian variational principle is equivalent to the Hamiltonian approach, as shown by Miles (1977). Note that the Hamiltonian $\mathcal{H}(\zeta, \partial_x \phi, \partial_z \phi)$ itself is the spatial integral of H:

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$\mathcal{H} = \int H \, \mathrm{d}x$. Now we make the following Ansatz for the potential $\phi(x,z,t)$, corresponding with a parabolic behaviour over depth with $\partial_z \phi = 0$ at the bed and $\phi = \varphi$ at the free surface:

$$\phi(x,z,t) = \varphi(x,t) \;+\; f(z;h,\zeta)\,\psi(x,t), \qquad \text{with} \tag{2.4}$$

$$f(z;h,\zeta) = \frac{1}{2}\,(\,z\,-\,\zeta\,)\,\frac{2\,h\,+\,z\,+\,\zeta}{h\,+\,\zeta}. \tag{2.5}$$

We take this choice, because we only want time derivatives of $\zeta(x,t)$ and $\varphi(x,t)$ to appear in the Euler-Lagrange equations, and because we know that for a horizontal bottom we have $\partial_z \phi = 0$ at $z = -h$ and therefor expect parabolic behaviour at leading order[a].

Under the assumption of a mildly sloping bottom, *i.e.* neglecting spatial derivatives of $h(x)$, the velocity components become:

$$\partial_x \phi \;=\; \partial_x \varphi \;-\; \frac{1}{2}\left[1\;+\;\left(\frac{h+z}{h+\zeta}\right)^2\right]\psi\,\partial_x\zeta \;+\; f(z;h,\zeta)\,\partial_x\psi \qquad \text{and} \qquad \partial_z\phi \;=\; \frac{h+z}{h+\zeta}\,\psi. \tag{2.6}$$

Note that $\psi(x,t)$ is the vertical velocity $\partial_z\phi$ at $z = \zeta(x,t)$. From these, we find for the energy density H:

$$\begin{aligned} H = {} & \frac{1}{2}\,(h+\zeta)\,\left[\partial_x\varphi \;-\; \frac{2}{3}\,\psi\,\partial_x\zeta \;-\; \frac{1}{3}\,(h+\zeta)\,\partial_x\psi\right]^2 \;+\; \frac{1}{90}\,(h+\zeta)\,\left[\psi\,\partial_x\zeta \;-\; (h+\zeta)\,\partial_x\psi\right]^2 \\ & + \frac{1}{6}\,(h+\zeta)\,\psi^2 \;+\; \frac{1}{2}\,g\,\zeta^2. \end{aligned} \tag{2.7}$$

Now by taking variations[b] of $\mathcal{L}$ with respect to φ, ζ and ψ we get from $\delta\mathcal{L} = 0$:

$$\partial_t\,\zeta + \partial_x\left\{(h+\zeta)\,\left[\partial_x\varphi \;-\; \frac{2}{3}\,\psi\,\partial_x\zeta \;-\; \frac{1}{3}\,(h+\zeta)\,\partial_x\psi\right]\right\} \;=\; 0, \tag{2.8}$$

$$\begin{aligned} & \partial_t\,\varphi + g\,\zeta \;+\; \frac{1}{2}\left[\partial_x\varphi \;-\; \frac{2}{3}\,\psi\,\partial_x\zeta \;-\; \frac{2}{3}\,(h+\zeta)\,\partial_x\psi\right]^2 \;-\; \frac{1}{45}\left[\psi\,\partial_x\zeta \;+\; (h+\zeta)\,\partial_x\psi\right]^2 \\ & + \frac{1}{6}\left[1+\frac{1}{5}\,(\partial_x\zeta)^2\right]\psi^2 \;+\; \partial_x\left\{(h+\zeta)\left[\frac{2}{3}\,\partial_x\varphi \;-\; \frac{7}{15}\,\psi\,\partial_x\zeta \;-\; \frac{1}{5}\,(h+\zeta)\,\partial_x\psi\right]\psi\right\} = 0, \end{aligned} \tag{2.9}$$

$$\begin{aligned} & (h+\zeta)\;\psi\,\left[\frac{1}{3}\;+\;\frac{7}{15}\,(\partial_x\zeta)^2\right] \;-\; \left[\frac{2}{3}\,(h+\zeta)\,\partial_x\varphi \;-\; \frac{1}{5}\,(h+\zeta)^2\,\partial_x\psi\right]\,\partial_x\zeta \\ & + \partial_x\left\{\frac{1}{3}\,(h+\zeta)^2\,\partial_x\varphi \;-\; \frac{1}{5}\,(h+\zeta)^2\,\psi\,\partial_x\zeta \;-\; \frac{2}{15}\,(h+\zeta)^3\,\partial_x\psi\right\} \;=\; 0. \end{aligned} \tag{2.10}$$

We introduce $u \equiv \partial_x\varphi$, and note from (2.8) that the discharge $q(x,t)$ and depth-averaged velocity $U(x,t)$ are:

$$q \;\equiv\; (h+\zeta)\,U, \qquad \text{and} \qquad U \;=\; u \;-\; \frac{2}{3}\,\psi\,\partial_x\zeta \;-\; \frac{1}{3}\,(h+\zeta)\,\partial_x\psi. \tag{2.11}$$

Then the system of equations to be solved can be written as:

$$\partial_t\,\zeta + \partial_x\;(\,(h+\zeta)\,U\,) \;=\; 0, \tag{2.12}$$

$$\begin{aligned} & \partial_t\,u + \partial_x\left\{g\,\zeta \;+\; \frac{1}{2}\left[U \;-\; \frac{1}{3}\,(h+\zeta)\,\partial_x\psi\right]^2 \;-\; \frac{1}{45}\,\left[\psi\,\partial_x\zeta \;+\; (h+\zeta)\,\partial_x\psi\right]^2\right. \\ & \left. + \frac{1}{6}\left[1\;+\;\frac{1}{5}\,(\partial_x\zeta)^2\right]\psi^2 \;+\; \partial_x\left[(h+\zeta)\,\left(\frac{2}{3}\,u \;-\; \frac{7}{15}\,\psi\,\partial_x\zeta \;-\; \frac{1}{5}\,(h+\zeta)\,\partial_x\psi\right)\,\psi\right]\right\} \;=\; 0, \end{aligned} \tag{2.13}$$

$$\begin{aligned} & (h+\zeta)\;\psi\,\left[\frac{1}{3}\;+\;\frac{7}{15}\,(\partial_x\zeta)^2\right] \;-\; \left[\frac{2}{3}\,(h+\zeta)\;u \;-\; \frac{1}{5}\,(h+\zeta)^2\,\partial_x\psi\right]\,\partial_x\zeta \\ & + \partial_x\left\{\frac{1}{3}\,(h+\zeta)^2\;u \;-\; \frac{1}{5}\,(h+\zeta)^2\,\psi\,\partial_x\zeta \;-\; \frac{2}{15}\,(h+\zeta)^3\,\partial_x\psi\right\} \;=\; 0. \end{aligned} \tag{2.14}$$

So we have to solve two time-evolution equations for $\zeta(x,t)$ and $u(x,t)$, as well as an elliptic equation[c] for $\psi(x,t)$. Further it can be observed that, for given $\zeta(x,t)$ and $\varphi(x,t)$, equation (2.14) is a linear equation in $\psi(x,t)$.

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[a] Higher-order performance can be obtained by chosing $\phi(x,z,t) = \varphi(x,t) + \sum_{m=1}^{M}(z-\zeta)^m\,\beta_m$. Chosing $M = 2$ also gives a parabolic approximation to $\phi(x,z,t)$, but results in better linear dispersion than the present model. The cost is that two elliptic equations for $\beta_1(x,t)$ and $\beta_2(x,t)$ have to be solved, instead of one elliptic equation for $\psi(x,t)$ in the present model.

[b] With $L = \varphi\,\partial_t\zeta - H\,(\zeta, \partial_x\zeta, \varphi, \partial_x\varphi, \psi, \partial_x\psi; x, t)$ we have $\delta\mathcal{L} = \iint\left(\delta\varphi\,\frac{\delta L}{\delta\varphi} + \delta\zeta\,\frac{\delta L}{\delta\zeta} + \delta\psi\,\frac{\delta L}{\delta\psi}\right)\mathrm{d}x\,\mathrm{d}t$ and *e.g.* $\frac{\delta L}{\delta\varphi} = \partial_t\zeta - \left[\frac{\partial H}{\partial\varphi} - \partial_x\left(\frac{\partial H}{\partial(\partial_x\varphi)}\right)\right]$.

[c] We talk already of *elliptic* in anticipation to the two-dimensional extension we are planning.

3. Weakly non-linear model. If we assume that also the free-surface slope $\partial_x \zeta$ is small and can be neglected in the Hamiltonian density (2.7), we get after taking the variations the following system of equations:

$$\partial_t \zeta + \partial_x \left\{ (h+\zeta) \left[u - \frac{1}{3} (h+\zeta)\, \partial_x \psi \right] \right\} = 0, \tag{3.1}$$

$$\partial_t u + \partial_x \left\{ g\,\zeta + \frac{1}{2} \left[u - \frac{2}{3} (h+\zeta)\, \partial_x \psi \right]^2 - \frac{1}{45} (h+\zeta)^2 (\partial_x \psi)^2 + \frac{1}{6} \psi^2 \right\} = 0, \tag{3.2}$$

$$(h+\zeta)\, \psi + \partial_x \left\{ (h+\zeta)^2\, u - \frac{2}{5} (h+\zeta)^3\, \partial_x \psi \right\} = 0. \tag{3.3}$$

These are somewhat simpler in appearance as the fully non-linear system (2.12)–(2.14). Note that this also has a positive-definite Hamiltonian.

4. Linear dispersion. When linearizing the equations for a horizontal bed, *i.e.* the still-water depth h is constant, we get:

$$\partial_t \zeta + h\, \partial_x u - \frac{1}{3} h^2\, \partial_x^2 \psi = 0, \qquad \partial_t u + g\, \partial_x \zeta = 0 \qquad \text{and} \qquad h\, \psi + h^2\, \partial_x u - \frac{2}{5} h^3\, \partial_x^2 \psi = 0. \tag{4.1}$$

We look for linear wave solutions propagating as $b(x,t) = \hat{b} \exp[\, i\,(kx - \omega t)\,]$, where k is the wave number and ω is the angular frequency. We find for the above linearized Boussinesq-type equations, with $(\hat{\zeta}, \hat{u}, \hat{\psi})$ denoting the complex-valued amplitudes of (ζ, u, ψ) respectively:

$$-\, i\, \omega\, \hat{\zeta} + i\, k\, h\, \hat{u} + \frac{1}{3} k^2\, h^2\, \hat{\psi} = 0, \qquad -i\, \omega\, \hat{u} + i\, g k\, \hat{\zeta} = 0 \qquad \text{and} \qquad h\, \hat{\psi} + i\, k\, h^2\, \hat{u} + \frac{2}{5} k^2\, h^3\, \hat{\psi} = 0. \tag{4.2}$$

Non-trivial solutions exist only if the following dispersion relationship is satisfied:

$$\frac{\omega^2\, h}{g} = (kh)^2\, \frac{1 + \frac{1}{15}\, (kh)^2}{1 + \frac{2}{5}\, (kh)^2}. \tag{4.3}$$

The first terms of a Taylor-series expansion around $kh = 0$ are:

$$\frac{\omega^2\, h}{g} = (kh)^2 - \frac{1}{3} (kh)^4 + \frac{2}{15} (kh)^6 - \frac{4}{75} (kh)^8 + \mathcal{O}\left((kh)^{10} \right). \tag{4.4}$$

This dispersion relation can be compared with the exact linear dispersion relation:

$$\omega^2 = g\, k\, \tanh\, kh, \tag{4.5}$$

which has the Taylor-series expansion

$$\frac{\omega^2\, h}{g} = (kh)^2 - \frac{1}{3} (kh)^4 + \frac{2}{15} (kh)^6 - \frac{17}{315} (kh)^8 + \mathcal{O}\left((kh)^{10} \right). \tag{4.6}$$

So they start differing with the term proportional to $(kh)^8$. Both dispersion curves are compared in Figure 5.1(a). They differ less than 1% for $kh < 2.3$ and less than 2.8% for $kh < \pi$.

5. Numerical examples. In order to test the models some preliminary computations on a horizontal bed are performed. As initial condition we use periodic waves computed with the method of Rienecker and Fenton (1981). We both test the fully non-linear (2.12)–(2.14)) and the weakly non-linear (3.1)–(3.3) model.

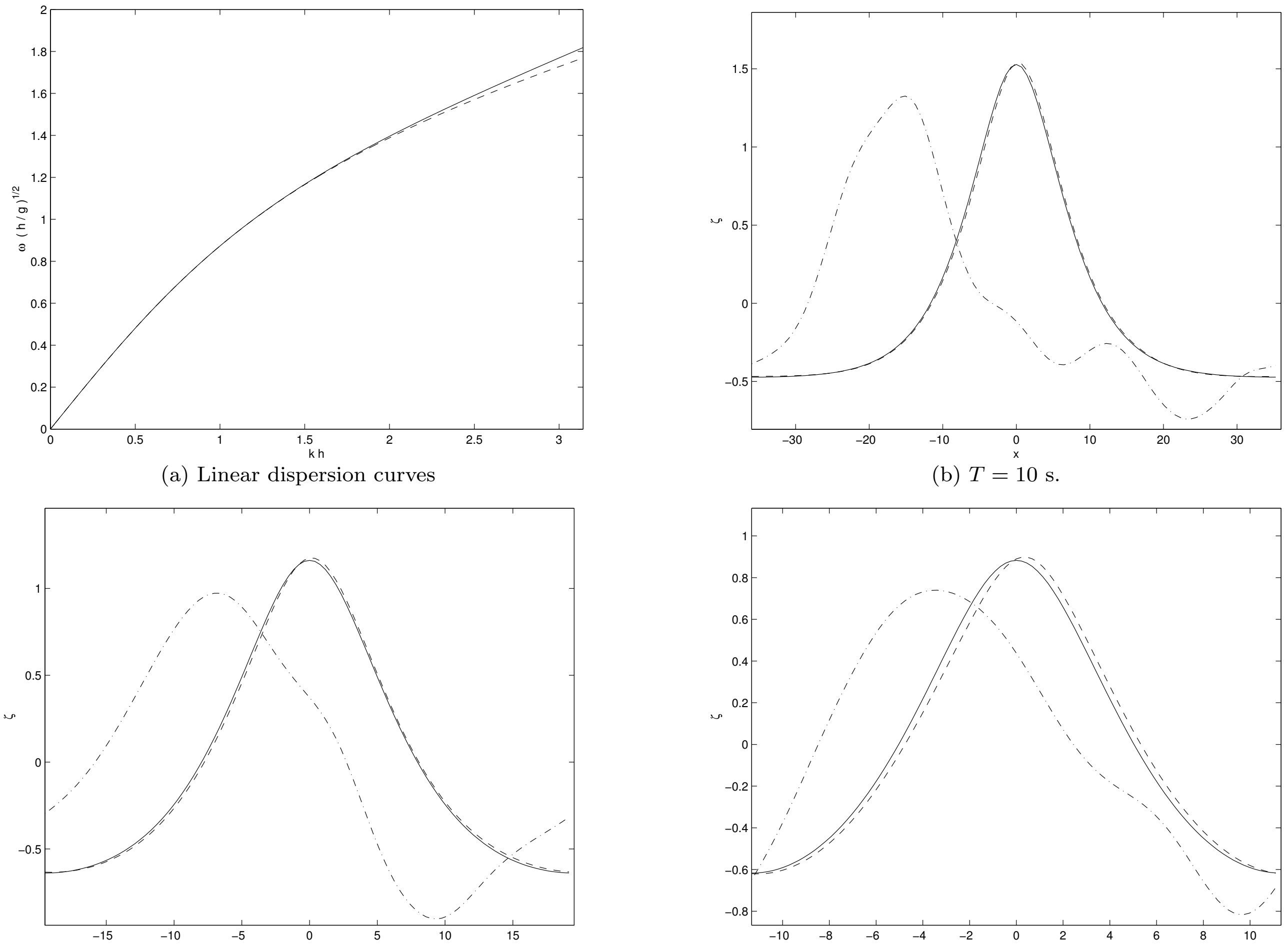
T [s]	H [m]	h/λ	H/h
10	2.0	0.0695	0.40
6	1.8	0.1280	0.36
4	1.5	0.2208	0.30

TABLE 5.1
Wave conditions for numerical examples.

In all cases we use a constant water depth $h = 5$ m and $g = 9.81$ m/s^2. We consider three cases as given in Table 5. A pseudo-spectral method has been used with 100 points per wave length and also 100 points per wave period. A periodic spatial domain has been used. The elliptic equation for ψ has been solved by means of a conjugate gradient method (the bi-CGSTAB method), see Quarteroni and Valli (1997). The time-integration has been performed using a four-stage Runge-Kutta integration method. No numerical damping and smoothing have been applied.

The results of the computations are shown after a simulation time of five wave periods, see Figure 5.1. During these computations no instabilities occurred. In a similar model using Agnon *et al.* (1999) we often encountered numerical instabilities. We think the present model performs well because of the positive definiteness of the Hamiltonian density, which guarantees good dynamical behaviour of the approximate equations. A check on the numerical accuracy has been performed. Averaging over one wave length shows that the absolute errors of the fully non-linear model and $T = 4$ s are as follows after five wave periods: for the Hamiltonian density $2 \cdot 10^{-5}$ with $\langle H \rangle = 2.6504$, for ζ the error is $3 \cdot 10^{-17}$ and for the free-surface potential gradient we find an error of $2 \cdot 10^{-16}$. Also for the weakly non-linear model the errors are of similar magnitude.

As is obvious from Figures 5.1(b-d) the weakly non-linear model performs not good enough for practical purposes. The fully non-linear model performs very well.



(a) Linear dispersion curves

(b) $T = 10$ s.

(c) $T = 6$ s.

(d) $T = 4$ s.

FIG. 5.1. *(a): Linear dispersion curves* $\omega\sqrt{h/g}$ *as a function of kh for the Boussinesq model (4.3) (solid line) and the exact linear dispersion (4.5) (dash-dash line). (b)–(d): Snapshots of the free-surface elevation after 5 wave periods (b)–(d). of the fully non-linear model (dash-dash line), the weakly non-linear model (dash-dot line) and the Rienecker and Fenton solution (solid line).*

6. Conclusion. We have presented a relatively easy derivation of Boussinesq-like equations from a variational principle having a positive definite Hamiltonian. The resulting mass and momentum equations have conservative form. Preliminary computations show promising behaviour of the fully non-linear model. At the conference we intend to show results compared with measurements of waves over varying bathymetry and with other models.

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