A nonlinear model for surface waves interacting with a surface-piercing cylinder

by John Grue

Mechanics Division, Department of Mathematics, University of Oslo, Norway

Interaction between steep water waves and fixed or moving bodies at the surface of the ocean represents continued interest from theoretical, experimental and practical point of view. Linear and weakly nonlinear wave-body interaction theories are well developed and implemented in computer codes for industrial use. The next step for the offshore industry is to include in their routines a set of formulations and codes that take into consideration the effects of strongly nonlinear waves interacting with geometries. This paper is a contribution towards that direction. We shall study a fully nonlinear formulation of an interaction between incoming waves and a vertical cylinder that is circular and is surface piercing. We here study the case of a fixed, non-moving cylinder. The formulation may be considered as a first step of a formulation of the flow in an outer domain which is matched to the flow in an inner domain. In this case more terms (or integrals) enter into the formulation. These integrals do not represent any new unknowns, however, and contribute thus to more complicated effects driving an outer flow, without involving essentially more complex analysis.

We study here the simple case of incoming waves in deep water interacting with a vertical, circular cylinder. A potential flow formulation is adopted where the fluid velocity is determined by the gradient of a velocity potential ϕ. The elevation of the moving free surface boundary is determined by η which is a function of the horizontal co-ordinate $\mathbf{x} = (x_1, x_2)$ and time t. As vertical co-ordinate y is introduced. For convenience we shall assume that the level $y = 0$ is above the maximal surface elevation. We shall denote the value of the potential at the free surface by $\widetilde{\phi}(\mathbf{x}, t) = \phi(\mathbf{x}, \eta(\mathbf{x}, t), t)$. The variables $\widetilde{\phi}$ and η can be integrated forward in time once the normal velocity ϕ_n at the free surface is known. A relation between ϕ_n and $(\widetilde{\phi}, \eta)$ is obtained from the solution of the Laplace equation in the fluid domain. A fully nonlinear formulation in the case when there is no geometry in the fluid (Clamond and Grue, 2001,§6, Grue, 2002) has recently been fully implemented and tested out, see Fructus et al. (2005), Clamond et al. (2005).

The equations become somewhat more involved when a geometry is present in the fluid. Solution of the Laplace equation is obtained using Green's theorem. The potential $\widetilde{\phi}$, surface elevation and normal velocity of the free surface are defined outside the cylinder. We shall assume that all these quantities are zero in the fraction of the horizontal plane inside the cylinder.

For a field point that is on the free surface the integral equation formulation gives:

$$\int_{S_F} \frac{V'}{R} d\mathbf{x}' = 2\pi\widetilde{\phi} + I_F(\widetilde{\phi}) + \int_{S_B} \phi'_B \frac{\partial}{\partial n'} \frac{1}{r} dS + 2\pi T(\widetilde{\phi}) + 2\pi N(V), \tag{1}$$

where

$$I_F(\widetilde{\phi}) = \int_{S_F} (\eta' - \eta)\nabla' \widetilde{\phi}' \cdot \nabla' \frac{1}{R} d\mathbf{x}' - \int_{L_F} \mathbf{n} \cdot [\widetilde{\phi}'(\eta' - \eta)\nabla \frac{1}{R}] dl', \tag{2}$$

$$T(\widetilde{\phi}) = \frac{1}{2\pi} \int_{S_F} \widetilde{\phi}' [1 - (1 + D^2)^{-3/2}] \nabla' \cdot [(\eta' - \eta)\nabla' \frac{1}{R}] d\mathbf{x}', \tag{3}$$

$$N(V) = \frac{1}{2\pi} \int_{S_F} \frac{V'}{R} [1 - (1 + D^2)^{-1/2}] d\mathbf{x}'. \tag{4}$$

Here, S_F denotes the free surface, L_F the intersection line between the free surface and the cylinder, and S_B the wetted body surface. Further, $V = \phi_n[1 + |\nabla\eta|^2]^{1/2}$. The distance between a source point and field point is r and the corresponding horizontal distance is $R = |\mathbf{x}' - \mathbf{x}|$. The difference in elevation at the field point and source point divided by the horizontal distance is denoted by $D = (\eta' - \eta)/R$. We note that $D \to 0$ for $R \to \infty$ and $D \to \eta_R$ for $R \to 0$. (A prime denotes the value at the field point, e.g. $\eta' = \eta(\mathbf{x}', t)$.) In the case without a body eq. (1) fits with the corresponding equation derived in Clamond and Grue (2001,§6), Grue (2002).

In the integral over S_F we employ that $\frac{1}{R} = \mathcal{F}^{-1}\{\frac{2\pi}{k}e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}\}$ where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ inverse transform. In the integral over S_B we employ a decomposition:

$$\frac{1}{r} = \frac{1}{r_0} - \eta\frac{\partial}{\partial y'}\frac{1}{r_0} + \frac{1}{\mathcal{R}}, \tag{5}$$

where r_0 denotes the value of r with $y = \eta = 0$ and is obtained by Fourier transform by

$$\frac{1}{r_0} = \mathcal{F}^{-1}\left\{\frac{2\pi}{k}e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'+ky'}\right\}, \qquad y' < 0. \tag{6}$$

Here $\mathbf{k} = (k_1, k_2)$ denotes the wavenumber vector, $k = |\mathbf{k}|$, and the level of $y = 0$ chosen above any level of the free surface. The contribution to eq. (1) from S_B is simplified introducing

$$\Phi_B(\mathbf{k}) = -\frac{\partial}{\partial K}\int_0^{2\pi}\int_{-\infty}^{0}\phi'_B e^{-\mathrm{i}K\cos(\alpha-\theta')+ky'}dy'ad\theta', \tag{7}$$

$$\Phi_{BL}(\mathbf{k}) = -\frac{\partial}{\partial K}\int_0^{2\pi}\int_{0}^{\eta_0}\phi'_B e^{-\mathrm{i}K\cos(\alpha-\theta')+ky'}dy'ad\theta', \tag{8}$$

$$\Phi_{BL0}(\mathbf{k}) = -\frac{\partial}{\partial K}\int_0^{2\pi}\int_{0}^{\eta_0}\phi'_B e^{-\mathrm{i}K\cos(\alpha-\theta')}dy'ad\theta', \tag{9}$$

where $\mathbf{k} = k(\cos\alpha, \sin\alpha)$, $\mathbf{x}' = a(\cos\theta', \sin\theta')$ on the vertical cylinder, where a denotes the cylinder radius, and $K = ka$. η_0 denotes the elevation along the cylinder. It is sufficient to evaluate $-\infty < \phi_B < 0$ to obtain $\Phi_B(\mathbf{k})$.

The integral over the body and the contribution due to $I_F(\widetilde{\phi})$ in eq. (2) sum up to

$$\begin{aligned} I_F(\widetilde{\phi}) + \int_{S_B}\phi'_B\frac{\partial}{\partial n'}\frac{1}{r}dS &= 2\pi\mathcal{F}^{-1}(\Phi_B) - 2\pi\eta\mathcal{F}^{-1}(k(\mathcal{F}(\widetilde{\phi}) + \Phi_B)) - 2\pi\mathrm{i}\mathcal{F}^{-1}\left(\frac{\mathbf{k}}{k}\cdot\mathcal{F}(\eta\nabla\widetilde{\phi})\right) \\ &+ 2\pi\mathcal{F}(\Phi_{BL} - \Phi_{BL0}) - 2\pi\eta\mathcal{F}^{-1}(k\Phi_{BL}) + \int_{S_B}\phi'_B\frac{\partial}{\partial n'}\frac{1}{\mathcal{R}}dS \end{aligned} \tag{10}$$

Applying Fourier transform to eq. (1) we obtain

$$\frac{\mathcal{F}(V)}{k} = \mathcal{F}(\widetilde{\phi}) + \Phi_B - \mathcal{F}(\eta\mathcal{F}^{-1}[k\mathcal{F}(\widetilde{\phi}) + k\Phi_B]) - \mathrm{i}\frac{\mathbf{k}}{k}\cdot\mathcal{F}(\eta\nabla\widetilde{\phi}) + \mathcal{F}(N(V) + T(\widetilde{\phi}) + T_B(\phi_B)) \tag{11}$$

where we have introduced

$$T_B(\phi_B) = \mathcal{F}^{-1}(\Phi_{BL} - \Phi_{BL0}) - \eta\mathcal{F}^{-1}(k\Phi_{BL}) + \frac{1}{2\pi}\int_{S_B}\phi'_B\frac{\partial}{\partial n'}\frac{1}{\mathcal{R}}dS \tag{12}$$

The eq. (11) is linear in V and a decomposition of the various contributions is suitable, i.e., $V = V_1 + V_2 + V_3 + V_4 + V_5$, where

$$\frac{\mathcal{F}(V_1)}{k} = \mathcal{F}(\widetilde{\phi}) + \Phi_B \tag{13}$$

$$\frac{\mathcal{F}(V_2)}{k} = -\mathcal{F}(\eta V_1) - \mathrm{i}\frac{\mathbf{k}}{k}\cdot\mathcal{F}(\eta\nabla\widetilde{\phi}) \tag{14}$$

$$\frac{\mathcal{F}(V_3)}{k} = \mathcal{F}(T(\widetilde{\phi})) \tag{15}$$

$$\frac{\mathcal{F}(V_4)}{k} = \mathcal{F}(N(V)) \tag{16}$$

$$\frac{\mathcal{F}(V_5)}{k} = \mathcal{F}(T_B(\phi_B)) \tag{17}$$

For the evaluation point on S_B we obtain

$$2\pi\phi_B + \int_{S_B} \phi'_B \frac{\partial}{\partial n'}\frac{1}{r} dS + \mathcal{I}(\widetilde{\phi}) + \mathcal{J}(V) = 0, \qquad \mathbf{x}' \text{ on } S_B \tag{18}$$

where we have introduced

$$\mathcal{I}(\widetilde{\phi}) = \int_{S_F} \widetilde{\phi}' \frac{\partial}{\partial n'}\frac{1}{r} dS, \qquad \mathcal{J}(V) = -\int_{S_F} \phi'_n \frac{1}{r} dS. \tag{19}$$

A decomposition is chosen by

$$\frac{1}{r} = \frac{1}{r_0} - \eta' \frac{\partial}{\partial y}\frac{1}{r_0} + \frac{1}{\mathcal{R}_2}, \tag{20}$$

where r_0 is the value of r with $y' = \eta' = 0$. For the integrals $\mathcal{I}$ and $\mathcal{J}$ we obtain

$$\begin{aligned}\mathcal{I}(\widetilde{\phi}) = {} & 2\pi\mathcal{F}^{-1}(e^{ky}[-\mathcal{F}(\widetilde{\phi}) - \mathrm{i}\frac{\mathbf{k}}{k}\cdot\mathcal{F}(\eta\nabla\widetilde{\phi}) + \frac{\partial}{\partial K}\int_{L_F} \widetilde{\phi}'\eta' e^{-\mathrm{i}K\cos(\alpha-\theta')} a d\theta']) \\ & + \int_{S_F} d\mathbf{x}'\widetilde{\phi}' \left[(-\nabla'\eta'\cdot\nabla')\left(\frac{1}{r} - \frac{1}{r_0}\right) + \frac{\partial}{\partial y}\frac{1}{\mathcal{R}_2}\right]\end{aligned} \tag{21}$$

$$\mathcal{J}(V) = 2\pi\mathcal{F}^{-1}\left\{\left(-\frac{\mathcal{F}(V)}{k} + \mathcal{F}(\eta V)\right)e^{ky'}\right\} - \int_{S_F} \frac{V'}{\mathcal{R}_2} dS \tag{22}$$

The equation for ϕ_B becomes

$$\begin{aligned}&\phi_B + \frac{1}{2\pi}\int_{S_B} \phi'_B \frac{\partial}{\partial n'}\frac{1}{r} dS + \mathcal{F}^{-1}\{e^{ky}\frac{\partial}{\partial K}\int_{L_F} \widetilde{\phi}'\eta' e^{-\mathrm{i}K\cos(\alpha-\theta')} a d\theta'\} \\ &+ \mathcal{F}^{-1}\{e^{ky}[-\mathcal{F}(\widetilde{\phi}) - \frac{\mathcal{F}(V)}{k} + \mathcal{F}(\eta V) - \mathrm{i}\frac{\mathbf{k}}{k}\cdot\mathcal{F}(\eta\nabla\widetilde{\phi})]\} \\ &+ \frac{1}{2\pi}\int_{S_F} d\mathbf{x}' \left\{\widetilde{\phi}'\left[\frac{\partial}{\partial y}\frac{1}{\mathcal{R}_2} - \nabla'\eta'\cdot\nabla'\left(\frac{1}{r} - \frac{1}{r_0}\right)\right] - \frac{V'}{\mathcal{R}_2}\right\} = 0\end{aligned} \tag{23}$$

The relation for $\mathcal{F}(V)/k$ is substituted into the equation above. We introduce the operator $\mathcal{L}(\phi_B)$ by

$$\begin{aligned}\mathcal{L}(\phi_B) = {} & \phi_B + \frac{1}{2\pi}\int_{S_B} \phi'_B \frac{\partial}{\partial n'}\frac{1}{r} dS + \mathcal{F}^{-1}\{e^{ky}[-\Phi_B + 2\mathcal{F}(\eta\mathcal{F}^{-1}[k\Phi_B])]\} \\ & + \mathcal{F}^{-1}\{e^{ky}\frac{\partial}{\partial K}\int_{L_F} \widetilde{\phi}'\eta' e^{-\mathrm{i}K\cos(\alpha-\theta')} a d\theta'\}\end{aligned} \tag{24}$$

and functions $H(\widetilde{\phi})$ and $H_3(V, \phi_B)$ by

$$\begin{aligned}-H(\widetilde{\phi}) = {} & \mathcal{F}^{-1}\{e^{ky}[-2\mathcal{F}(\widetilde{\phi}) + 2\mathcal{F}(\eta\mathcal{F}^{-1}[k\mathcal{F}(\widetilde{\phi})]) - \mathcal{F}(T(\widetilde{\phi}))\} \\ & + \frac{1}{2\pi}\int_{S_F} d\mathbf{x}'\widetilde{\phi}'\left\{\frac{\partial}{\partial y}\frac{1}{\mathcal{R}_2} - \nabla'\eta'\cdot\nabla'\left(\frac{1}{r} - \frac{1}{r_0}\right)\right\},\end{aligned} \tag{25}$$

$$-H_3(V, \phi_B) = \mathcal{F}^{-1}\{e^{ky}[\mathcal{F}(\eta(V - V_1)) - \mathcal{F}(N(V)) - \mathcal{F}(T_B(\phi_B))]\} - \frac{1}{2\pi}\int_{S_F} d\mathbf{x}'\frac{V'}{\mathcal{R}_2}. \tag{26}$$

The equation for ϕ_B may then be written on the form $\mathcal{L}(\phi_B) = H(\widetilde{\phi}) + H_3(V, \phi_B)$. The two unknown quantities (V and ϕ_B) are obtained by a procedure of successive approximations. First we decompose the potential at the body by $\phi_B = \phi_{B,1} + \phi_{B,3}$ where $\phi_{B,1}$ is a leading order contribution and $\phi_{B,3}$ a relatively small contribution (of cubic order in appearance). The quantity Φ_B is decomposed accordingly, i.e. $\Phi_B = \Phi_{B,1} + \Phi_{B,3}$. The first component (the leading order) of ϕ_B is obtained by $\phi_{B,1} = \mathcal{L}^{-1}(H(\widetilde{\phi}))$. This is obtained explicitly since the potential $\widetilde{\phi}$ at the free surface is a given input to the algoritm to find V and ϕ_B. V is then obtained successively by

$$\frac{\mathcal{F}(V_{1,1})}{k} = \mathcal{F}(\widetilde{\phi}) + \Phi_{B,1} \tag{27}$$

$$\frac{\mathcal{F}(V_{2,1})}{k} = -\mathcal{F}(\eta V_{1,1}) - \mathrm{i}\frac{\mathbf{k}}{k} \cdot \mathcal{F}(\eta \nabla \widetilde{\phi}) \tag{28}$$

$$\phi_{B,3} = \mathcal{L}^{-1}(H_3(V_{1,1} + V_{2,1}, \phi_{B,1})) \tag{29}$$

$$\frac{\mathcal{F}(V_{1,3})}{k} = \Phi_{B,3} \tag{30}$$

$$\frac{\mathcal{F}(V_{2,3})}{k} = -\mathcal{F}(\eta V_{1,3}) \tag{31}$$

$$\frac{\mathcal{F}(V_{4,1})}{k} = \mathcal{F}(N(V_{1,1} + V_{2,1})) \tag{32}$$

$$\frac{\mathcal{F}(V_{5,1})}{k} = \mathcal{F}(T_B(\phi_{B,1})) \tag{33}$$

The sum $V_{1,1} + V_{2,1} + V_3 + V_{4,1} + V_{5,1} + V_{1,3} + V_{2,3}$ approximates V to a very high accuracy. Further iterations may improve the accuracy in very steep (nonbreaking) waves.

¡

The funding by the Research Council of Norway through the NFR146526/420 Strategic University Programme *Modeling of currents and waves for sea structures* is gratefully acknowledged.

References

D. Clamond and J. Grue. A fast method for fully nonlinear water wave computations. *J. Fluid Mech.* 2001, Vol. 447.

J. Grue. On four highly nonlinear phenomena in wave theory and marine hydrodynamics. *Applied Ocean Res.*, 2002, Vol 24.

D. Fructus, D. Clamond, J. Grue and Ø. Kristiansen. An efficient model for three-dimensional surface wave simulations. Part I Free space problems. *J. Computational Physics.* 2005 (in press).

D. Clamond, D. Fructus, J. Grue and Ø. Kristiansen. An efficient model for three-dimensional surface wave simulations. Part II Generation and absorption. *J. Computational Physics.* 2005 (in press).