TWO-DIMENSIONAL ANALYSIS OF THE WAVE EXCITING FORCE ON A MULTI-HULL STRUCTURE

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INTRODUCTION

We presented an analysis of the hydro-elastic behavior of a very large mobile offshore structure [1] at the last workshop, which have a flexible broad deck supported by number of slender hulls, and we found that the motion in beam seas is one of important issue for the designing of this kind of structure. We treat the problem as a two-dimensional problem in this paper, since it is a good simplification for representing the motion of slender hulls in beam seas. We suppose a two-dimensional floating structure, which has a elastic deck supported by equally spaced demi-hulls. This is one of classical two-dimensional multi-hull interaction problems. However, it seems that there is no significant finding on the two-dimensional multi-hull interaction problem since an early work done by Ohkusu [2]. We try to make some improvements on this problem.

ZEROS OF WAVE EXCITING FORCE

Suppose equally spaced N demi-hulls supporting a flexible deck, and regular incident waves are coming from left to right. All demi-hulls have the same shape, and the width is much smaller than the space between two demi-hulls. It is well known that the wide spacing approximation works very well in this case. Applying the wide spacing approximation, the following relations are obtained for traveling waves in the water surface between adjacent demi-hulls

$$\begin{cases} \eta_n^+ = \eta_{n+1}^- C_R e^{-iK\ell} + \eta_{n-1}^+ C_T e^{-iK\ell}, \\ \eta_n^- = \eta_{n+1}^- C_T e^{-iK\ell} + \eta_{n-1}^+ C_R e^{-iK\ell}, \end{cases} \tag{1}$$

where, $\eta_n^\pm$ denotes the amplitude of traveling waves from the n-th demi-hull to positive or negative direction, C_R and C_T represent the reflection coefficient and the transmission coefficient respectively, K denotes the wave number of the traveling waves and ℓ denotes the space between adjacent demi-hulls. The numbering of the demi-hulls is made so that the first hull is located at the left end and the N-th hull is at the right end. It is noted that there is no wave due to the motion of demi-hulls in Eq.(1), since we only consider the diffraction problem. Although the 0-th nor the $N+1$-th hull do not exist, the boundary conditions for Eq.(1) are given as follows:

$$\eta_0^+ = 1 \quad \text{and} \quad \eta_{N+1}^- = 0. \tag{2}$$

It is convenient to represent the elastic motion of the deck as a summation of the modal functions $\psi_n(x)$. Analytical solutions of the modal function for Euler's beam with the free-free boundary condition are well known as

$$\psi_0(x) = 1, \qquad \psi_1(x) = \sqrt{12}\left(\frac{x}{B} - \frac{1}{2}\right), \tag{3}$$

$$\psi_j(x) = \frac{\cosh p_j B - \cos p_j B}{\sinh p_j B - \sin p_j B}\left(\sinh p_j x + \sin p_j x\right) + \cosh p_j x - \cos p_j x, \tag{4}$$

where, B denotes the overall length of the beam and p_j represents the eigen-values. It is mentioned that ψ_1 and ψ_2 are the heave mode and the roll mode respectively, and from the third to the ∞ modes are called vibration modes. Applying the orthogonality of ψ_j, the system of equations of motions is decomposed into a system of equations of modal amplitudes in which modal exciting forces due to diffraction waves are given by

$$E_j = e_2 \sum_{n=1}^{N} \left(\eta_{n-1}^+ + \eta_{n+1}^-\right) \psi_j((n-1)\ell), \qquad (j = 0, 1, 2, \cdots), \tag{5}$$

where, e_2 denotes the wave exciting force in the heaving direction. It is noted that the moment acting on the beam at the connection point between the demi-hull and the beam is ignored for the shake of simplicity in this analysis. In addition, it is assumed that the wavelength of the incident wave is much longer than the width of the demi-hull. Thus, the reflection coefficient is very small $C_R = O(\varepsilon)$, where $\varepsilon \ll 1$. The energy conservation law suggests that $\Im[C_T] = O(\varepsilon)$ and $1 - \Re[C_T] = O(\varepsilon^2)$.

Appling the regular perturbation method, modal wave exciting forces may be given as a power series of ε

$$E_j = \sum_{m=0}^{\infty} \varepsilon^m E_j^{(m)}. \tag{6}$$

In the case of the heave motion, it is very easy to obtain the 0-th order solution

$$E_0^{(0)} = e_2 \sum_{n=1}^{N} e^{-inK\ell} = e_2 \frac{\sin \frac{N}{2} K\ell}{\sin \frac{Kl}{2}} e^{-i\frac{K\ell}{2}(N+1)}. \tag{7}$$

(7) gives a classical knowledge that zeros of Froude-Krylov force are located at

$$K\ell = 2\frac{n}{N}\pi, \quad (n = 1, 2, 3 \cdots). \tag{8}$$

In order to get the first order solution, we neglect higher order terms in (1), and recurrence formulae for $\eta_n^{\pm}$ are obtained. Solving the formulae, $\eta_n^{\pm}$ is given as

$$\eta_n^{+} = e^{-inKl}\{1 + n(C_T - 1)\} + O(\varepsilon^2) \tag{9}$$

$$\eta_n^{-} = C_R \frac{\sin (N - n + 1)K\ell}{\sin K\ell} e^{-iNK\ell} + O(\varepsilon^2). \tag{10}$$

The first order solution of the heave exciting force is obtained by summing up the wave forces due to these waves

$$E_0^{(1)} = e_2 \frac{C_R}{\varepsilon} \frac{\sin \frac{N}{2} K\ell \sin \frac{N+1}{2} K\ell}{\sin \frac{Kl}{2} \sin K\ell} e^{-iNK\ell} + e_2 \frac{C_T - 1}{\varepsilon} \frac{1}{2 \sin \frac{K\ell}{2}} \left\{ i(N+1)e^{-i\left(N+\frac{1}{2}\right)K\ell} - \frac{1 - e^{-i(N+1)K\ell}}{2 \sin \frac{K\ell}{2}} \right\} \tag{11}$$

Since the first order solution is not simple, it is difficult to find zeros of the first order solution by analytical way. However, if you consider a small correction of zeros of 0-th order solution, the problem becomes very simple.

The zeros of the complete solution may exist near zeros of the 0-th order solution as

$$K\ell = 2\frac{n}{N}\pi + \delta, \quad \delta = O(\varepsilon) \qquad (n = 1, 2, 3 \cdots). \tag{12}$$

Substituting (12) into (7) and (11) and eliminating higher order terms, the heave exciting force near a zero of the 0-th order solution is obtained

$$E_0 = e_2 \frac{N}{2 \sin \frac{n}{N}\pi} e^{-i\frac{n}{N}\pi} \left(\delta - \Im[C_T]\right) + O(\varepsilon^2). \tag{13}$$

(13) suggests that the correction, which gives the improved zero, is given by $\delta = \Im[C_T]$.

The 0-th order solution for the roll motion is also obtained easily, however the solution is not simple

$$E_1^{(0)} = e_2 \frac{\sqrt{3}i}{(N-1)\sin \frac{K\ell}{2}} \left(N \cos \frac{N}{2} K\ell - \sin \frac{K\ell}{2} \cot \frac{K\ell}{2} \right) e^{-i\frac{K\ell}{2}(N+1)}. \tag{14}$$

The 0-th order solution for vibration modes are obtained as

$$\begin{aligned} E_j^{(0)} = \frac{1}{\sinh p_j x + \sin p_j x} \Bigg[& \frac{i e^{ip_j\ell}\left(e^{-iNp_j\ell} - e^{-iKN\ell}\right) - \sqrt{2}\left(1 - e^{-i(K-p_j)N\ell}\right) \sinh\left((N-1)p_j\ell + i\frac{\pi}{4}\right)}{2(e^{iK\ell} - e^{ip_j\ell})} \\ & - \frac{i e^{-ip_j\ell}\left(e^{iNp_j\ell} - e^{-iKN\ell}\right) + \sqrt{2}\left(1 - e^{-i(K+p_j)N\ell}\right) \sinh\left((N-1)p_j\ell - i\frac{\pi}{4}\right)}{2(e^{iK\ell} - e^{-ip_j\ell})} \\ & + \frac{e^{p_j\ell}\left(e^{-Np_j\ell} - e^{-iKN\ell}\right) - \sqrt{2}\left(1 - e^{-i(K+ip_j)N\ell}\right) \cos\left((N-1)p_j\ell + \frac{\pi}{4}\right)}{2(e^{iK\ell} - e^{p_j\ell})} \\ & - \frac{e^{-p_j\ell}\left(e^{Np_j\ell} - e^{-iKN\ell}\right) + \sqrt{2}\left(1 - e^{-i(K-ip_j)N\ell}\right) \cos\left((N-1)p_j\ell - \frac{\pi}{4}\right)}{2(e^{iK\ell} - e^{-p_j\ell})} \Bigg]. \end{aligned} \tag{15}$$

It is almost impossible to get zeros of (14) or (15) by analytical way, however the zeros are obtained by numerical computation accurately and quickly. The first order solution of the roll motion and vibration modes may be possible to obtain but it is not easy. However, our need is only to know zeros of the exciting forces, and there is another simple way to find them.

Let us consider Taylor's series expansion of the exciting forces near a zero of the force as we have done the same thing to obtain zeros of the heave exciting force

$$E_j(K_n\ell+\delta)=\delta\frac{dE_j^{(0)}(K_n\ell)}{d(K\ell)}+\varepsilon E_j^{(1)}O(\varepsilon^2), \tag{16}$$

where $K_n\ell$ is a zero of the 0-th order solution. Substituting (9) and (10) into (5), and eliminating the higher order terms, the right hand side of (16) is replaced by

$$E_j(K_n\ell+\delta)=i\left(\Im[C_T]-\delta\right)\sum_{n=1}^{N} ne^{-iK\ell}\psi_j((n-1)\ell)+O(\varepsilon^2). \tag{17}$$

It is mentioned that the first order exciting force due to η_n^- has the same zero as the 0-th order exciting force has, since the denominator of (10) is decomposed into the exponential form. (17) suggests that the small correction $\delta=\Im[C_T]$ is worth for improving zeros of all modal exciting forces.

Table.1 Zeros of the heave exciting force and the two-nodes vibration exciting force, where $N=10$ and $b/\ell=0.03$.

(a) Heave exciting force

Exact value	$K_n\ell$	$K_n\ell+\Im[C_T]$
0.666	0.628	0.665
1.325	1.257	1.324
1.977	1.885	1.978
2.613	2.513	2.628
3.091	3.142	3.270
3.959	3.770	3.918
4.577	4.398	4.558
5.188	5.027	5.195

(b) Two-nodes vibration mode

Exact value	$K_n\ell$	$K_n\ell+\Im[C_T]$
0.229	0.215	0.228
1.267	1.201	1.266
1.946	1.855	1.947
2.600	2.500	2.614
3.091	3.142	3.270
3.973	3.783	3.932
4.606	4.428	4.588
5.241	5.082	5.251

Table 1 shows zeros of the heave exciting force and the two-nodes vibration exciting force. It is apparent that the improved zero is in good agreement with the exact value.

NEAR RESONANCE

It is well known that resonance happens at certain frequencies in the case of an infinite periodic demi-hulls. Since the geometry has periodicity of ℓ, we may represent waves as

$$\eta_n^{\pm}=e^{-i\beta\ell n}\mathcal{A}^{\pm}, \tag{18}$$

where $\mathcal{A}^{\pm}$ is a constant. (18) expresses that there is a change in phase of $e^{-i\beta\ell}$ from the wave between the $n-1$-th hull and the n-th hull to the wave between the n-th hull and the $n+1$-th hull. Substituting (18) into (1), a system of linear equation is obtained

$$\begin{cases}\left(1-C_Te^{i\beta\ell}\right)\mathcal{A}^+-C_Re^{-i\beta\ell}\mathcal{A}^-=0,\\ C_Re^{i\beta\ell}\mathcal{A}^+-\left(1-C_Te^{-i\beta\ell}\right)\mathcal{A}^-=0.\end{cases} \tag{19}$$

If the determinant of (19) is zero

$$(C_R^2-C_T^2)e^{-2iK\ell}+2C_Te^{-iK\ell}\cos\beta\ell-1=0, \tag{20}$$

a resonance occurs in the system. If $\beta\ell=\pi$, the wave motion in the adjacent space is opposite. We may call this mode the Neumann mode from an analogy of the Rayleigh-Bloch waves along periodic gratings, and we may call the case $\beta\ell=2\pi$ the Dirichlet mode.

Evans and Porter [3] presented a good idea to explain the near-trapping by a finite linear array, in which they used a modulation of Rayleigh-Bloach waves. The same idea can be applied in the present problem. If we assume $\beta\ell=2\pi\pm\varepsilon$, which presents a small modulation of the Diriclet mode, roots of (20) are obtained as

$$e^{-iK\ell}=\frac{1+\dfrac{\varepsilon^2}{2}\dfrac{C_T}{C_R}}{C_T+C_R}\quad\text{or}\quad\frac{1-\dfrac{\varepsilon^2}{2}\dfrac{C_T}{C_R}}{C_T-C_R}+O(\varepsilon^3). \tag{21}$$

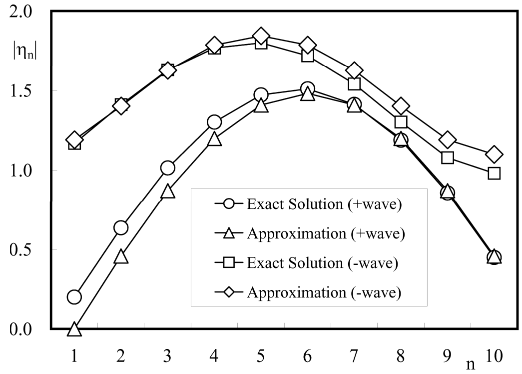

Figure 1: Comparison of $\eta_n^{\pm}$ between the exact solution and the approximation, where $N = 10$ $K\ell = 7.002$, which corresponds to the first root of the Dirichlet mode

Evans and Porter [3] showed that ε gives the shape of trapping mode, and if we assume $\varepsilon = \pi/N$, the mode may be a half wavelength of the sine shape. They suggested the existence of another mode, however we use the sine shape for the first trial. The wave system may be presented as

$$\eta_n^+ = a^+ e^{i\varepsilon n} + b^+ e^{-i\varepsilon n}, \tag{22}$$

$$\eta_n^- = a^- e^{i\varepsilon n - 1} + b^- e^{-i\varepsilon(n-1)}. \tag{23}$$

Boundary conditions (2) gives the following relations

$$b^+ = 1 - a^+ \quad \text{and} \quad b^- = -a^-. \tag{24}$$

Substituting (22) and (23) into (1) and considering cases of $n = 0$ and $n = 1$, wave amplitudes are obtained as

$$a^- = \frac{i}{2\sin\varepsilon}\frac{C_R}{C_T}, \qquad a^+ = \frac{i}{2}\left\{ e^{-i\varepsilon} - \frac{1}{C_T}(C_R^2 - C_T^2)e^{-iK\ell} \right\}. \tag{25}$$

If you substitute the first root of (21) and eliminate higher order terms, the amplitude of positive directional wave is represented as

$$a^+ = i\frac{e^{-i\varepsilon\frac{C_T}{C_R}}}{2\sin\varepsilon}\frac{C_R}{C_T} + O(\varepsilon^3). \tag{26}$$

Combining (25) and (26), a near resonance mode of waves among demi-hulls is obtained

$$\eta_n^+ = -\frac{C_R}{C_T}\frac{\sin\left(n - \frac{C_T}{C_R}\right)\varepsilon}{\sin\varepsilon}, \qquad \eta_n^- = -\frac{C_R}{C_T}\frac{\sin(n-1)\varepsilon}{\sin\varepsilon}. \tag{27}$$

Figure 1 shows a comparison of $\eta_n^{\pm}$ between the exact solution and the approximation, where $N = 10$ $K\ell = 7.002$, which corresponds to the first root of the Dirichlet mode. The approximation is in good agreement with the exact solution, although the number of demi-hulls is only ten. The agreement is improved as the number of demi-hulls increases.

References

[1] Takagi, K and Yano, W : Analysis of Hydroelastic Behavior of a Very Large Mobile Offshore Structure in Waves, 18th IWWWFB, Le Crosic, France, 2003

[2] Ohkusu, M : The analysis of wave forces acting upon multi-float-supported platforms, Trans. of West Japan Soc. of Naval Architects, vol. 51, 153-170, 1976.

[3] Evans, DV and Porter, R : Trapping and near-trapping by arrays of cylinders in waves, J of Eng. Math. 35, 149-179 1999.

Discusser: M. McIver

In the table of the zeros of the heave exciting force, there is one value at about 3.1 where your approximation makes the result go further from the exact result. Can you explain this?

Author's reply:

There is a resonant frequency near this one. Therefore, the assumption of small reflection is violated.

Discusser: R. Porter

Your analysis is not confined to the particular system you look at in your paper; it applies to any multi-scattering problem where there exist a periodic array of finite scatterer when you know the reflection and transmission coefficients. The system you describe is therefore very similar to those found in, for example, the theory of crystal lattices and is closely related to Bragg resonance. A good reference is G.A. Kreigsmann, QJMAM (2003), title " ...Fabry-Perot resonator ...".

Author's reply:

Thank you for your comment.