Wave scattering by an ice sheet of variable thickness

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Introduction

This paper describes an investigation into the effect on wave propagation of an ice sheet of varying thickness floating on water of varying depth. A variational principle equivalent to the governing equations of linear theory in three dimensions is given, forming the basis of a Rayleigh-Ritz approximation. Here we invoke the mild-slope approximation in respect of the ice thickness and water depth variations to derive a relatively simple model for slowly modulating waves from which the vertical coordinate is absent. We consider both the scattering of flexural-gravity waves by variations in the thickness of an infinite ice sheet and the depth variations, and the scattering of free surface gravity waves by an ice sheet of finite extent and varying thickness, with arbitrary topography again incorporated.

Existing work related to the present contribution includes that of Meylan (2001) who also used a variational principle for the full linearised equations to investigate wave interaction with rectangular plates of constant thickness in three dimensions. Andrianov & Hermans (2003) recently derived an integro-differential equation for the problem of an sheet of finite extent floating in a free surface.

Formulation and approximation

Using cartesian coordinates x, y, z with z directed vertically upwards, $z = -h(x, y)$ coincides with the bed and $z = -d(x, y)$ with the lower surface of the ice sheet, which is represented by an elastic plate of constant density ρ_i and varying thickness $D(x, y)$.

The usual assumptions of linear wave wave theory and the removal of a harmonic time dependence lead to the equations

$$\nabla^2\phi = 0 \quad (-h < z < -d), \qquad \phi_z + \nabla_h h.\nabla_h\phi = 0 \quad (z = -h), \tag{1}$$

representing conservation of mass in the fluid and zero normal flow across the bed; $\phi(x, y, z)\exp(-\mathrm{i}\omega t)$ is the velocity potential of the fluid motion and the notation

$$\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z), \quad \nabla_h = (\partial/\partial x, \partial/\partial y, 0)$$

has been adopted. Using thin plate theory and representing the elevation of the ice-fluid interface from its equilibrium position by $\eta(x, y)\exp(-\mathrm{i}\omega t)$, we obtain

$$(1-\alpha)\eta + \mathcal{L}\eta - \phi = 0, \quad \nabla_h d.\nabla_h\phi + \phi_z = \kappa\eta \quad (z = -d), \tag{2}$$

which respectively represent the equation of motion of the plate and the kinematic condition at the interface. The operator $\mathcal{L}$ is defined by

$$\mathcal{L}\eta \equiv \nabla_h^2(\beta\nabla_h^2\eta) - (1-\nu)\{\beta_{xx}\eta_{yy} + \beta_{yy}\eta_{xx} - 2\beta_{xy}\eta_{xy}\}$$

and

$$\kappa = \omega^2/g, \quad \alpha(x, y) = \kappa\rho_i D(x, y)/\rho_w, \quad \beta(x, y) = ED^3(x, y)/12\rho_w g(1-\nu^2),$$

where ν is Poisson's ratio for ice, $\rho_w g\beta$ is its flexural rigidity, E is Young's modulus and ρ_w is the density of water.

The functions $\phi(x, y, z)$ and $\eta(x, y)$ are given by solving (1) and (2), together with far field and edge conditions as appropriate. Alternatively, these equations may be regarded as the natural conditions of the variational principle $\delta L = 0$, where L is the functional defined by

$$\begin{aligned} L(\psi, \chi) \quad = \quad & \frac{1}{2}\iint_{\mathcal{D}}\Big\{\int_{-h}^{-d}(\nabla\psi)^2\,\mathrm{d}z - 2\kappa\beta(1-\nu)(\chi_{xx}\chi_{yy} - \chi_{xy}^2) \\ & \qquad +\kappa\{(1-\alpha)\chi^2 + \beta(\nabla_h^2\chi)^2 - 2\chi[\psi]_{-d}\}\Big\}\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

$\mathcal{D}$ being a simply connected domain in the plane $z = 0$. Because the integrand has the value of the Lagrangian density at (x, y), $\delta L = 0$ is in effect Hamilton's principle. It can be verified that L is indeed stationary at $\psi = \phi$ and $\chi = \eta$, where ϕ and η satisfy (1) and (2); the natural conditions at the boundary of $\mathcal{D}$ are not relevant to the investigation.

It has been tacitly assumed so far that D is twice continuously differentiable. Where this is not the case (1) and (2) are replaced by jump conditions and these can be deduced from the variational principle by introducing an internal boundary, Γ say, in $\mathcal{D}$. If $\mathbf{n} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$, $\mathbf{s} = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$ are local unit vectors respectively normal and tangential to Γ, where $\theta = \theta(s)$ and s measures arc length on Γ, then the additional natural conditions of $\delta L = 0$ are continuity across $\Gamma \times [-h, -d]$ of ϕ_n and across Γ of

$$\left.\begin{aligned} \mathcal{M}\eta &\equiv \nabla_h^2\eta - (1-\nu)(\eta_{ss} + \theta'\eta_n), \\ \mathcal{S}\eta &\equiv (\beta\nabla_h^2\eta)_n - (1-\nu)\{(\eta_{ss} + \theta'\eta_n)\beta_n - 2(\eta_{ns} - \theta'\eta_s)\beta_s - (\eta_{ns} - \theta'\eta_s)_s\beta\}, \end{aligned}\right\} \tag{3}$$

which are the bending moment and shear stress. Here, $\partial/\partial n = \mathbf{n}.\nabla_h$ and $\partial/\partial s = \mathbf{s}.\nabla_h$. If Γ is a line of discontinuity of D, then $\mathcal{M}\eta$ and $\mathcal{S}\eta$ are required to vanish on each side of Γ.

The Rayleigh-Ritz method may be applied to the variational principle: an approximation to its stationary point is also an approximation to its set of natural conditions (1), (2) and, if appropriate, (3). Solutions of these equations can therefore be obtained to any accuracy by taking a large enough trial space.

Here we take the one-term trial function

$$\phi(x, y, z) \approx \psi(x, y, z) = \varphi(x, y)w(x, y, z), \quad w(x, y, z) = \operatorname{sech} k(h-d)\cosh k(z+h), \tag{4}$$

where $k = k(x, y)$ denotes the positive real root of the local dispersion relation

$$(1 - \alpha + \beta k^4)k \tanh k(h-d) = \kappa \tag{5}$$

with $h = h(x, y)$, $d = d(x, y)$ and $D = D(x, y)$. The motivation for this choice is that, with h, d and D constant, $\exp(\pm \mathrm{i}kx)\operatorname{sech} k(h-d)\cosh k(z+h)$ are plane wave solutions of (1) and (2), k also being constant in this case. We therefore expect φ to represent waves modulated by the variations in h, d and D in the general case.

Implementing $\delta L = 0$ with ψ given by (4), we find that

$$\nabla_h.a\nabla_h\varphi + b\varphi + \kappa\chi = 0, \quad (1-\alpha)\chi + \mathcal{L}\chi - \varphi = 0, \tag{6}$$

are the resulting approximations of (1) and (2), where

$$\left.\begin{aligned} a &= \int_{-h}^{-d} w^2 \,\mathrm{d}z = (4k)^{-1}\operatorname{sech}^2(kH)\{2kH + \sinh(2kH)\}, \\ b &= k^2 a - k\tanh(kH) + \nabla_h.\int_{-h}^{-d} w\nabla_h w \,\mathrm{d}z - \int_{-h}^{-d} (\nabla_h w)^2 \,\mathrm{d}z, \end{aligned}\right\} \tag{7}$$

in which $H(x, y) = h(x, y) - d(x, y)$. We note from (5) that $k \equiv k(H, D)$ and it is easily shown from (7) that $a \equiv a(H, D)$ and $b \equiv b(H, D)$. Thus the equations (6) that determine the approximations $\phi \approx \varphi w$ and $\eta \approx \chi$ contain the geometry of the problem only through $H(x, y)$ and $D(x, y)$. This is to be expected as we have combined the averaging that leads to thin plate theory with averaging through the fluid layer. We remark that the so-called "mild-slope approximation" that we have invoked to derive (6) is an extension of the shallow water approximation to the whole wavelength régime; the shallow water counterpart of (6) is obtained by taking the long wave limit in the first equation.

Discontinuity conditions that must replace (6) where H and D are not sufficiently differentiable can be deduced from the variational principle. In particular, continuity of $\mathcal{M}\chi$ and $\mathcal{S}\chi$ applies to $\chi \approx \eta$.

Two-dimensional scattering

We illustrate the approximation by applying it in a two-dimensional context where $h = h(x)$, $d = d(x)$, $D = D(x)$ and the motion is independent of y. In this case (6) may be written as the second order system

$$(a(x)\phi_0')' + b(x)\phi_0 + \kappa\phi_1 = 0, \quad \beta(x)\phi_1'' - \phi_2 = 0, \quad \phi_2'' + (1-\alpha(x))\phi_1 - \phi_0 = 0, \tag{8}$$

where $\phi_0 = \varphi$, $\phi_1 = \chi$ and $\phi_2 = \beta\phi_1''$ is the bending moment.

We suppose that

$$\left.\begin{array}{l} h(x) = h_0, \quad d(x) = d_0, \quad D(x) = D_0 \quad (x < 0) \\ h(x) = h_1, \quad d(x) = d_1, \quad D(x) = D_1 \quad (x > \ell) \end{array}\right\}, \tag{9}$$

where h_i, d_i and D_i are constants for $i = 0, 1$. It is not difficult to show that in regions where $h = h_i$, $d = d_i$ and $D = D_i$ a complete set of solutions $\mathbf{\Phi} = (\phi_0, \phi_1, \phi_2)^T$ of (8) is

$$\mathbf{c}_i(k_i)\mathrm{e}^{\pm\mathrm{i}k_i x}, \quad \mathbf{c}_i(\mu_i)\mathrm{e}^{\pm\mathrm{i}\mu_i x}, \quad \mathbf{c}_i(\overline{\mu}_i)\mathrm{e}^{\pm\mathrm{i}\overline{\mu}_i x},$$

with

$$\mathbf{c}_i(u) = (1, \kappa^{-1}(a_i u^2 - b_i), -\kappa^{-1}\beta_i u^2(a_i u^2 - b_i))^T.$$

Here k_i denotes the positive real root of (5) with $H = H_i$ and $D = D_i$, and $\mu_i = p_i + \mathrm{i}q_i$, where $p_i > 0$ and $q_i > 0$. For each of the subdomains $x < 0$ and $x > \ell$, $\mathbf{\Phi}$ therefore consists of an incoming and an outgoing propagating wave mode and two exponentially decaying modes. Explicitly,

$$\mathbf{\Phi}(x) = C_0(A_0\mathrm{e}^{\mathrm{i}k_0 x}, 0, 0)^T + C_0(B_0\mathrm{e}^{-\mathrm{i}k_0 x}, B_0^{(1)}\mathrm{e}^{-\mathrm{i}\mu_0 x}, B_0^{(2)}\mathrm{e}^{\mathrm{i}\overline{\mu}_0 x})^T \quad (x < 0), \tag{10}$$

and

$$\mathbf{\Phi}(x) = C_1(A_1\mathrm{e}^{\mathrm{i}k_1(\ell-x)}, 0, 0)^T + C_1(B_1\mathrm{e}^{-\mathrm{i}k_1(\ell-x)}, B_1^{(1)}\mathrm{e}^{-\mathrm{i}\mu_1(\ell-x)}, B_1^{(2)}\mathrm{e}^{\mathrm{i}\overline{\mu}_1(\ell-x)})^T \quad (x > \ell), \tag{11}$$

where C_i denotes the 3×3 matrix $C_i = (\mathbf{c}_i(k_i), \mathbf{c}_i(\mu_i), \mathbf{c}_i(\overline{\mu}_i))$. The amplitudes $B_i^{(j)}$ $(i = 0, 1; j = 1, 2)$ of the evanescent waves are unknown and the assigned incident wave amplitudes A_0 and A_1 and the unknown scattered wave amplitudes B_0 and B_1 can be connected by

$$\begin{pmatrix} B_0 \\ B_1 \end{pmatrix} = S \begin{pmatrix} A_0 \\ A_1 \end{pmatrix}, \quad S = \begin{pmatrix} R_0 & T_1 \\ T_0 & R_1 \end{pmatrix}, \tag{12}$$

where R_i, T_i are the reflection and transmission coefficients for a wave incident from $x = -\infty$ $(i = 0)$ and $x = \infty$ $(i = 1)$.

The solutions (10) and (11) provide boundary conditions for a numerical solution of (8) in the interval $0 < x < \ell$ by using appropriate continuity or jump conditions at $x = 0$ and $x = \ell$.

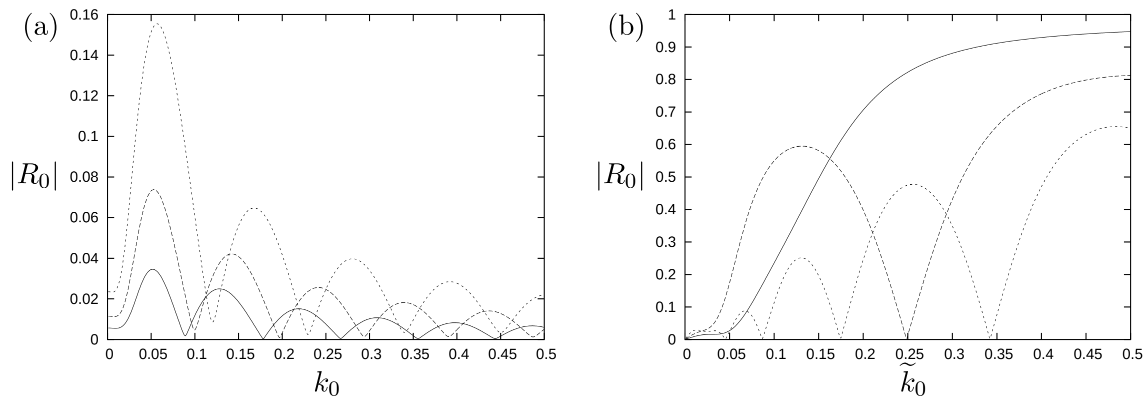
Partial ice cover

For the case in which the ice sheet does not extend to infinity, the variational principle has to be amended to include domains corresponding to an unloaded free surface. The approximation corresponding to (4) can be used in such domains and is simply

$$\phi(x, y, z) \approx \psi(x, y, z) = \varphi(x, y)w(x, y, z), \quad w(x, y, z) = \operatorname{sech}(\widetilde{k}h)\cosh\widetilde{k}(z + h),$$

where $\widetilde{k} = \widetilde{k}(x, y)$ denotes the positive real root of the usual dispersion relation $\widetilde{k}\tanh(\widetilde{k}h) = \kappa$. The single equation corresponding to (6) in the free surface regions is the modified mild-slope equation and (6) continues to apply unchanged where there is ice cover.

In two dimensions, we can consider the situation given in (9) with $d_i = D_i = 0$ for $i = 0, 1$, so that a free surface occupies $x < 0$ and $x > \ell$. In the semi-infinite regions there are now only travelling wave modes with wavenumbers $\widetilde{k}_0$ for $x < 0$ and $\widetilde{k}_1$ for $x > \ell$ and the conditions matching the solutions in these regions with the numerical solutions at $x = 0$ and $x = \ell$ include the vanishing of the moment and the shear stress at the edge of the ice. With these adjustments, the structure has the same form as that considered above for the case of overall ice cover and, in particular, (12) may still be used to describe the scattering process.



Numerical results

Numerical results for two-dimensional scattering are shown in the figure for the two problems described above. The parameter values used to produce the graphs are $E = 5\text{GPa}$, $\nu = 0.3$, $\rho_w = 1025\text{kgm}^{-3}$ and $\rho_i = 922.5\text{kgm}^{-3}$. We take $d = \rho_i D/\rho_w$ and apply Archimedes' principle for finite sheets.

In part (a) of the figure we give an example of scattering over a flat bed by a linearly thickening ice sheet with $D(x) = D_0 + A_D x/\ell$ $(0 < x < \ell)$. The magnitude of the reflection coefficient R_0 is plotted against the wavenumber of the incident wave for $D_0 = 1\text{m}$, $H_0 = 20\text{m}$, $\ell = 40\text{m}$ and $A_D = 0.5\text{m}$ (solid line), 1m (long dash), 2m (short dash). As expected, interference effects between the ends of the varying ice region dominate the graph and the maximum reflected energy increases with the ice thickness.

In part (b) a corresponding set of results is shown for a finite ice sheet of parabolic profile

$$D(x) = D_0 + 4A_D(1 - x/\ell)(x/\ell) \quad (0 < x < \ell).$$

The parameter measured along the abscissa is again the wavenumber of the (free surface) incident wave and $D_0 = 0.1\text{m}$, $H_0 = 20\text{m}$, $A_D = 1.35\text{m}$ so that the ice sheet is 14.5 times thicker at its centre than at its ends. The graphs show the effect of varying sheet length, with $\ell = 40\text{m}$ (solid line), 80m (short dash) and 160m (long dash). The proportion of energy reflected decreases as the length of the sheet increases for the same maximum thickness, suggesting that the gradient of the sheet is a key factor in the scattering process.

For a finite sheet of constant thickness, the results that we obtain are in good agreement with those of other authors such as Tkacheva (2002). Comparison with the experimental data of Utsunomiya *et al* (1995) for the bending moment and shear stress in the sheet is also reasonably good. In the case of varying sheet and bed geometries, the accuracy of our model can be analysed by returning to the variational principle and using a higher order approximation based on finitely many evanescent wave modes.

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