Linear Time-Dependent Motion of a Two Dimensional Floating Elastic Plate in Finite Depth Water using the Laplace Transform

Michael H. Meylan
Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealan
email: meylan@math.auckland.ac.nz

Christophe Hazard and Francois Loret
Laboratoire de Propagation des Ondes : Etude Mathématique et Simulation
(UMR 2706 du CNRS): ENSTA / POems, 32 Boulevard Victor, 75015 Paris, France
email:hazard@ensta.fr and loret@ensta.fr

ABSTRACT

We solve the linear problem of a elastic plate of negligible draft floating on the surface of water of finite depth in the time domain using the Laplace transform. We show the solution can be found by using the free-surface Green function defined for complex values of the frequency. The results obtained using the Laplace transform are show to agree with those found using an eigenfunction expansion method.

Introduction

Linear water wave theory, which still remains the principle design tool in offshore engineering, is often regarded as classical and largely mature. Most of the relevant theory can be found in Stoker (1957) and Wehausen & Laitone (1960) which were written over forty years ago. This is especially true of the single frequency solution, however it is less true of the time-dependent problem which has not received the same research attention. The time-dependent problem has been neglected for two reason, firstly the single frequency solution is often all that is required for engineering design, and secondly because the time-dependent problem is another order of magnitude in difficulty (since it involves temporal and well as spatial derivatives).

Two solution methods are presently in Stoker (1957) and Wehausen & Laitone (1960) to the time-dependent problem. The first is to use the so called memory effect function and the second is to use the time-dependent Green function. Both of these methods suffer from significant draw backs. Recently an alternative method to solve the time-dependent problem has been developed using an eigenfunction expansion method by Hazard & Lenoir (2002), Hazard & Loret (2004), Meylan (2002) and Hazard & Meylan (2004).

In the present paper we consider solving the time-dependent problem using the Laplace transform. The Laplace transform, as well as providing a solution to the time-dependent problem is important as a starting point in the development of the singularity expansion method Hazard (2000). The singularity expansion method represents the solution as an expansion in the scattering frequencies plus some possible additional terms. We apply this method to the specific problem of a floating elastic plate. The floating elastic plate is chosen because of the experience the authors have in solving this problem, because this problem is of physical significance as a model for large floating structures and because this problem seems likely to have significant scattering frequencies.

The Laplace transform solution method we present depends on calculating the free-surface Green function for complex frequencies. We discuss this issue in some detail. The solution method is not in anyway limited to the problem of a floating plate and could easily be applied to other linear wave problems, for example a rigid barrier or a rigid floating body.

Thin Plate Problem

We begin with the equations of motion. The mathematical description of the problem follows from Stoker (1957) and it also described in Meylan (2002) for the case of shallow water. The water occupies the region $-\infty < x < \infty$ and $-h < z < 0$ and a thin elastic plate of negligible draft floats on the water surface $-b < z < b$. The velocity potential of the water, Φ, satisfies Laplace's equation and the no flow condition at the bottom

$$\Delta\Phi = 0, \quad -h < z < 0,$$
$$\partial_n \Phi = 0, \quad z = -h.$$

At the free surface the kinematic condition applies,

$$\partial_t \zeta = \partial_n \Phi, \quad z = 0,$$

where ζ is the displacement of the water surface or the plate (from the shallow draft approximation) and ∂_n is the outward normal derivative. The dynamic condition obtained by matching the pressure also applies at the free surface,

$$-\rho g\zeta-\rho\partial_t\Phi=\left\{\begin{array}{ll}0, & x\notin(-b,b),\\ D\partial_x^4\zeta+\rho' d\,\partial_t^2\zeta, & x\in(-b,b),\end{array}\right.\quad z=0.$$

where D is the bending rigidity of the plate per unit length, ρ is the density of water, ρ' is the density of the plate, d is the plate thickness and g is the acceleration due to gravity. At the ends of the plate the free edge boundary conditions

$$\lim_{x\downarrow -b}\partial_x^2\zeta=\lim_{x\uparrow b}\partial_x^2\zeta=\lim_{x\downarrow -b}\partial_x^3\zeta=\lim_{x\uparrow b}\partial_x^3\zeta=0$$

are applied since these are common in offshore engineering applications (Namba & Ohkusu (1999)). However the theory which will be developed applies equally to any of the energy-conserving edge conditions such as clamped or pinned and there is no need for the boundary conditions to be symmetric.

Non-dimensional variables are now introduced using a length parameter L for the space variables (which may be chosen as the water depth or the characteristic length $(D/\rho g)^{1/4}$) and $\sqrt{L/g}$ for the time variable:

$$\bar{x}=\frac{x}{L},\quad \bar{z}=\frac{z}{L}\quad\text{and}\quad \bar{t}=\frac{t}{\sqrt{L/g}}.$$

Hence the non-dimensional surface displacement and velocity potential, defined by

$$\bar{\zeta}(\bar{x},\bar{t})=\frac{\zeta(x,t)}{L}\quad\text{and}\quad\bar{\Phi}(\bar{x},\bar{z},\bar{t})=\frac{\Phi(x,z,t)}{L\sqrt{gL}},$$

satisfy the following coupled equations

$$\begin{gathered}\bar{\Delta}\bar{\Phi}=0,\quad -\bar{h}<\bar{z}<0,\\ \partial_{\bar{n}}\bar{\Phi}=0,\quad \bar{z}=-\bar{h},\\ -\bar{\zeta}-\partial_{\bar{t}}\bar{\Phi}=\left\{\begin{array}{ll}0, & \bar{x}\notin(-\bar{b},\bar{b}),\\ \beta\partial_{\bar{x}}^4\bar{\zeta}+\gamma\partial_{\bar{t}}^2\bar{\zeta}, & \bar{x}\in(-\bar{b},\bar{b}),\end{array}\right.\quad \bar{z}=0,\\ \partial_{\bar{t}}\bar{\zeta}=\partial_{\bar{n}}\bar{\Phi},\quad \bar{z}=0,\end{gathered}\tag{1}$$

plus the free edge boundary conditions

$$\lim_{\bar{x}\downarrow -\bar{b}}\partial_{\bar{x}}^2\bar{\zeta}=\lim_{\bar{x}\uparrow \bar{b}}\partial_{\bar{x}}^2\bar{\zeta}=\lim_{\bar{x}\downarrow -\bar{b}}\partial_{\bar{x}}^3\bar{\zeta}=\lim_{\bar{x}\uparrow \bar{b}}\partial_{\bar{x}}^3\bar{\zeta}=0,\tag{2}$$

where

$$\beta=\frac{D}{\rho g L^4}\quad\text{and}\quad\gamma=\frac{\rho' d}{\rho L}.$$

For clarity the overbar is dropped from now on.

Laplace Transform

Now we take the Laplace transform, given by

$$\hat{\Phi}=\int_0^\infty\Phi(t)\,e^{-pt}dt$$

of equation (1) by and we obtain

$$\begin{gathered}\Delta\hat{\Phi}=0,\quad -h<z<0,\\ \partial_n\hat{\Phi}=0,\quad z=-h,\\ -\hat{\zeta}-p\hat{\Phi}+\Phi_0=\\ \left\{\begin{array}{ll}0, & x\notin(-b,b),\\ \beta\partial_{\bar{x}}^4\hat{\zeta}+\gamma\left(p^2\hat{\zeta}-p\zeta_0-\partial_t\zeta_0\right), & x\in(-b,b),\end{array}\right.\quad z=0,\\ p\hat{\zeta}-\zeta_0=\partial_n\hat{\Phi},\quad z=0,\end{gathered}\tag{3}$$

where

$$\Phi_0=\Phi|_{t=0},\quad \zeta_0=\zeta|_{t=0},\quad\text{and}\quad \partial_t\zeta_0=\partial_t\zeta|_{t=0}$$

are the initial conditions. Of course Φ_0 and $\partial_t\zeta_0$ are not independent and are related by a Dirihclet to Neuman operator. We can rewrite this equation as the following

$$\begin{gathered}\Delta\hat{\Phi}=0,\quad -h<z<0,\\ \partial_n\hat{\Phi}=0,\quad z=-h,\\ \hat{\zeta}=-\tfrac{p}{\beta}\int g(x,y)\ \hat{\Phi}(y)\,dy\\ +\tfrac{1}{\beta}\int g(x,y)\ \left(\Phi_0(y)+\gamma\left(p\zeta_0(y)+\partial_t\zeta_0(y)\right)\right)dy\quad z=0,\\ p\hat{\zeta}-\zeta_0=\partial_n\hat{\Phi},\quad z=0,\end{gathered}\tag{4}$$

where $g(x,y)$ the Green function which satisfies

$$\partial_{\bar{x}}^4 g+\frac{\left(\gamma p^2+1\right)}{\beta}g=\delta(x-y).$$

We can solve equation (4) with the Green function which satisfies

$$\begin{gathered}\Delta G=0,\quad -h<z<0,\\ \partial_n G=0,\quad z=-h,\\ G_n+p^2G=-\delta(x-y),\quad z=0,\end{gathered}$$

This analogous to the standard free-surface Green function except that consider complex values for p with positive real part. We use this Green function in Green's theorem which states that

$$\int_{-b}^{b}G_n\Phi-G\Phi_n=0.$$

This implies that

$$\Phi=\int_{-b}^{b}G\left(-p^2\Phi-p\bar{\zeta}+\zeta_0\right)$$

$$=\int_{-b}^{b}G\left(-p^2\Phi-p\left(-\frac{p}{\beta}\int_{-b}^{b}g(x,y)\ \hat{\Phi}(y)\,dy\right)\right)$$

$$+\int_{-b}^{b}G\left(p\left(\frac{1}{\beta}\int_{-b}^{b}g(x,y)\ \left(\Phi_0(y)\right)dy\right)+\zeta_0\right)$$

$$+\int_{-b}^{b}G\left(p\left(\frac{1}{\beta}\int_{-b}^{b}g(x,y)\ \left(\gamma\left(p\zeta_0(y)+\partial_t\zeta_0(y)\right)\right)dy\right)\right)$$

which can be rewritten as

$$\Phi=\int_{-b}^{b}G\left(-p^2\Phi+\frac{p^2}{\beta}\int_{-b}^{b}g(x,y)\ \hat{\Phi}(y)\,dy\right)+f\tag{5}$$

where

$$f = +\int_{-b}^{b} G\zeta_0$$
$$-p\int_{-b}^{b} G\left(\frac{1}{\beta}\int_{-b}^{b} g(x,y)\ \left(\Phi_0(y)\right)dy\right)$$
$$-p\int_{-b}^{b} G\left(\frac{1}{\beta}\int_{-b}^{b} g(x,y)\ \left(\gamma\left(p\zeta_0(y)+\partial_t\zeta_0(y)\right)\right)dy\right)$$

Calculating the Green function

The solution of the time-dependent problem using the Laplace transform which we have presented requires that we calculate the Green function for complex values of the parameter p. We can easily calculate the Green function assuming that $p \in \mathbb{R}e^+$ and we obtain

$$G(\mathbf{x},\mathbf{y}) = -\frac{1}{\pi}\int_0^\infty \frac{\cos(\omega|\mathbf{x}-\mathbf{y}|)}{p^2+\omega\tanh(\omega h)}d\omega. \tag{6}$$

We can also write this Green's function as the sum over the modes

$$G(\mathbf{x},\mathbf{y}) = -i\sum_{m=0}^{\infty}\frac{e^{ik_m|\mathbf{x}-\mathbf{y}|}}{\tanh(k_m H)+k_m H\operatorname{sech}^2(k_m h)} \tag{7}$$

where k_m are the positive imaginary roots of the dispersion equation

$$p^2 = \omega\tanh(\omega h) = 0 \tag{8}$$

We require the analytic extension of this Green function for the inverse Laplace transform. We calculate the analytic extension using the expression for the Green function given by equation (7). We solve for the roots of the dispersion equation k_m for a real value of p. We then use a homotopy method in which the value of p is slowly changed and the roots are recalculated using the old values as an initial guess. We find the in the limit as the $\mathbb{R}e\ (p) \to 0$ we obtain the standard free surface Green function given by

$$G(\mathbf{x},\mathbf{y}) = -i\sum_{m=0}^{\infty}\frac{e^{ik_m|\mathbf{x}-\mathbf{y}|}}{\tanh(k_m H)+k_m H\operatorname{sech}^2(k_m H)} \tag{9}$$

where k_0 is now the real root of $p^2+\omega\tanh(\omega H)=0$ and k_i, $i \geq 1$ are the real roots (remembering that $p = i\omega$ where ω is the real frequency. We can also continue the analytic extension in the left hand side of the complex plane (this is equivalent to extending the resolvent through the continuous spectrum). Sometimes the left hand plane is referred to as the unphysical domain because the outgoing wave solutions grow rather than decay as $x \to \infty$. This is the singularity expansion method Hazard (2000) that allows the solution to be expressed as a sum over the scattering frequencies plus a possible continuous contribution. Calculating the analytic extension of the finite depth Green function is described in Hazard & Lenoir (1993).

In the case of infinite depth the Green function is given by

$$G(\mathbf{x},\mathbf{y}) = -\frac{1}{\pi}\int_0^\infty \frac{\cos(\omega|\mathbf{x}-\mathbf{y}|)}{p^2+\omega}d\omega \tag{10}$$

and the analytic extension is much simpler. However, if we compare our finite depth Green function with the infinite depth Green function for the case of water which may be approximate as infinitely deep we find that there is no longer agreement between the Green functions in the left hand complex plane.

Results

We can solve equation (5) using a discretisation of the potential and converting the integral equation to a matrix equation. To calculate the time-dependent solution we must compute the inverse Laplace transform given by the formula

$$\Phi(\mathbf{x},t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a-i\infty}\Phi(\mathbf{x},p)\,e^{pt}dp \tag{11}$$

where a is a positive real number. While the inverse Laplace transform requires the solution of equation (5) for many values of p, once these have been calculated the solution to Φ to be found for all t using equation (11)

We calculate the response of the plate to an initial displacement at rest. The initial conditions are

$$\Phi_0 = \ \partial_t\zeta_0 = 0$$

and

$$\zeta_0 = \exp\left(-x^2/350\right)$$

with the $\beta = 2e4$, $\gamma = 0$ and $b = 50$ (these values being the same as those in Meylan (2002)). Figure 1 shows the displacement for the times $t = 0$, 10, 20, 30, 40, 50, 60, 70, and 80. The solid line is for $H = 1$, the dashed line for $H = 5$, the chained line for $H = 20$ and the dotted line is for $H = 100$. Figure 2 is the equivalent figure calculated using the eigenfunction expansion method Hazard & Meylan (2004). We can see that the two figures give similar results.

There are a number of truncation and discretisation parameters in the Laplace transform solution which effect the accuracy of the solution numerically. In figure 1 we have set the upper limit of the integration to be $\pi/2$ and we set spacing to be $\pi/200$. We have set the value of the real parameter in the inverse Laplace transform to be $a = 0.0001$.

Summary

We have considered the linear time-dependent problem of a floating elastic plate of negligible draft on water of finite depth. A solution method to this problem has been presented using the Laplace transform and a free surface Green function which is analogous to the

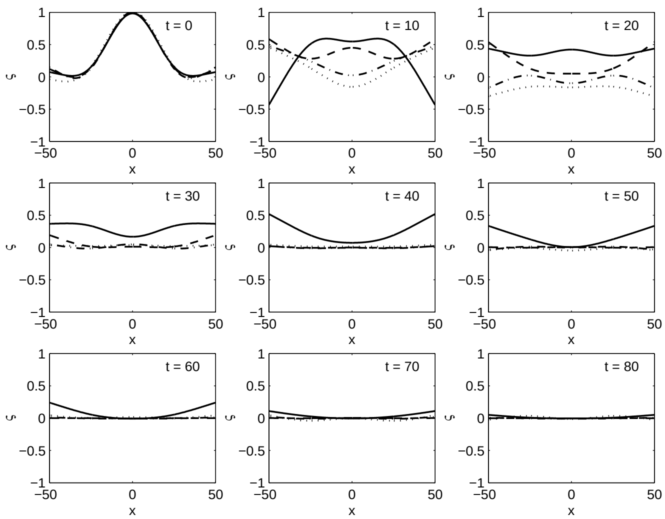

Fig. 1: The evolution of a symmetric displacement for the times shown. The plate occupies the region $-50 \leq x \leq 50$, $\beta = 2 \times 10^4$, $b = 50$. The solid line is for $H = 1$, the dashed line for $H = 5$, the chained line for $H = 20$ and the dotted line is for $H = 100$.. The Laplace transform method is used to make the calculations

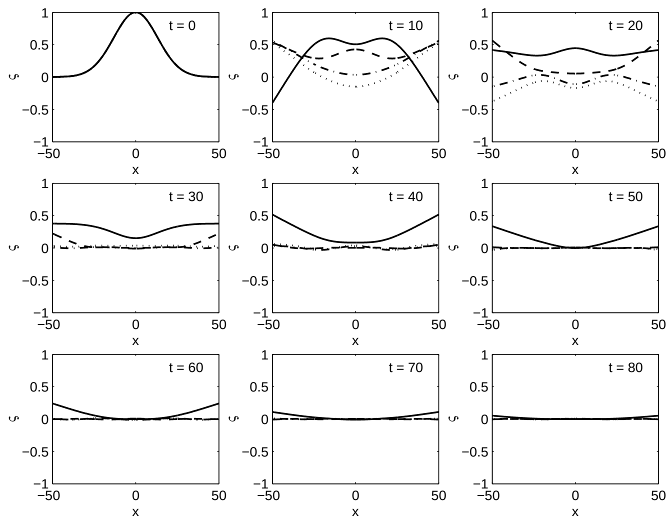

Fig. 2: The evolution of a symmetric displacement for the times shown. The plate occupies the region $-50 \leq x \leq 50$, $\beta = 2 \times 10^4$, $b = 50$. The solid line is for $H = 1$, the dashed line for $H = 5$, the chained line for $H = 20$ and the dotted line is for $H = 100$. The eigenfunction expansion method is used to make the calculations

standard free surface Green function except it is defined for complex values of the frequency parameter. We have shown that the results agree with those calculate by a different method. The Laplace transform solution method is in no way limited to problems which involve elastic plates and this work may be considered a simple example to demonstrate the method. A major challenge remains to apply the singularity expansion method and to express the solution as the sum over modes associated with the scattering frequencies (or resonances).

REFERENCES

Hazard, C. 2000 The singularity expansion method: an application in hydrodynamics. *Fifth International Conference on Mathematical and Numerical Aspects of Wave Propagation, Santiago de Compostela, Spain, July 2000* .

Hazard, C. & Lenoir, M. 1993 Determination of scattering frequencies for an elastic floating body. *SIAM J. Math. Anal.* **24** (4), 1458–1514.

Hazard, C. & Lenoir, M. 2002 Surface water waves. In *Scattering* (ed. eds. R. Pike & P. Sabatier), pp. 618–636. Academic.

Hazard, C. & Loret, F. 2004 Generalized eigenfunction expansions for scattering problem: application to water waves. *in preparation* .

Hazard, C. & Meylan, M. H. 2004 Spectral theory for a two-dimensional elastic thin plate floating on water of finite depth. *Journal of Fluid Mechanics* p. in preparation.

Meylan, M. H. 2002 Spectral solution of time dependent shallow water hydroelasticity. *J. of Fluid Mechanics.* **454**, 387 – 402.

Namba, Y. & Ohkusu, M. 1999 Hydroelastic behaviour of floating artifical islands in waves. *Int. J. of Offshore and Polar Engineering* **9** (2), 39–47.

Stoker, J. J. 1957 *Water Waves: The Mathematical Theory with Applications*. Interscience.

Wehausen, J. & Laitone, E. 1960 Surface waves. In *Fluid Dynamics III* (ed. S. Flügge & C. Truesdell), *Handbuch der Physik*, vol. 9, chap. 3, pp. 446–778. Springer Verlag.