Secularity and third-order wave scattering

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Introduction

The natural periods of many deep-water platforms fall in the range 3-5 s, and model tests and experiments have shown that resonance behaviour of such platforms occur in sea-states with peak periods which are typically 3 to 5 times their natural periods ([1], [3]). This nonlinear behaviour is known as 'ringing', and there has been much effort in recent years to describe the phenomenon mathematically either by extending second-order potential theory to higher order ([4], [7]) or by assuming that the body is slender compared to the wavelength [2].

The extension of the Stokes' expansion of the potential to third order, described in [4] and [7], is based on a regular perturbation expansion of the potential, and both papers concentrate on the third harmonic component of the potential, as this is the term which gives rise to the ringing phenomenon. However it is well-known ([6], §13.13) that secularity occurs at third-order in the regular perturbation expansion of the velocity potential for a propagating wave, and a strained coordinate expansion shows that this may be eliminated by allowing the dispersion relation to depend on the wave amplitude.

The situation is not quite so simple when there is a body in the fluid, as a wave which is incident on a structure is diffracted. In two dimensions this produces a reflected and transmitted wave in addition to the incident wave, and the purpose of the present work is to consider the secular terms that arise at third order as a result of these diifferent waves, and to look at ways of eliminating secularity. As will be shown below, nonlinear interactions means that it is not possible to modify the dispersion relation for each of the incident, reflected and transmitted waves as though they were in isolation. It will also be shown that the interaction of some of the terms in the first- and second- order potentials produces terms which are secular at third-order.

Formulation

A plane progressive wave is incident from the left on a fixed two-dimensional body in deep water, as illustrated in Figure 1. Carteisan coordiates (x, z) are chosen so that the origin is at the level

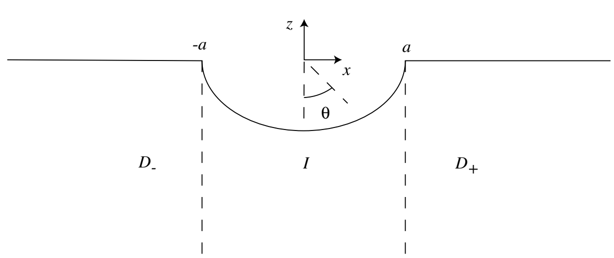

Figure 1: Definition sketch

of the first-order mean free surface and the z-axis points vertically upwards. The full nonlinear boundary value problem for the velocity potential $\Phi(x, z, t)$is given by

$$\nabla^2\Phi = 0 \quad \text{in the fluid,} \tag{1}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on the body,} \tag{2}$$

$$\nabla\phi \to 0 \quad \text{as } z \to -\infty, \tag{3}$$

$$\frac{\partial\Phi}{\partial z} - \frac{\partial\zeta}{\partial t} - \frac{\partial\Phi}{\partial x}\frac{\partial\zeta}{\partial x} = 0 \quad \text{on the free surface} \quad z = \zeta(x,t), \tag{4}$$

$$\frac{\partial\Phi}{\partial t} + g\zeta + \frac{1}{2}(\nabla\Phi)^2 = 0 \quad \text{on} \quad z = \zeta(x,t), \tag{5}$$

where g is the acceleration due to gravity. In addition a radiation condition must be prescribed. The kinematic and dynamic boundary conditions (4) and (5) may be combined together to give

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} = -2\nabla\Phi.\nabla\frac{\partial\Phi}{\partial t} - \frac{1}{2}\nabla\Phi.\nabla[(\nabla\Phi)^2] \quad \text{on} \quad z = \zeta(x,t). \tag{6}$$

A regular perturbation expansion for Φ

The underlying assumption of the Stokes' expansion of the velocity potential is that the incident wave amplitude A is small compared to the wavelength and body dimensions. The incident wave is assumed to have fundamental angular frequency ω, and a first-order, infinite depth wave number is defined by

$$k_1 = \omega^2/g. \tag{7}$$

A regular perturbation expansion of the velocity potential and wave elevation is sought in terms of powers of k_1A. Thus, it is convenient to write

$$\Phi = A\Phi_1 + A^2\Phi_2 + A^3\Phi_3 + O((k_1A)^4) \tag{8}$$

and

$$\zeta = A\zeta_1 + A^2\zeta_2 + A^3\zeta_3 + O((k_1A)^4). \tag{9}$$

The first-order potential is written as

$$\Phi_1 = \mathrm{Re}\left[-\frac{\mathrm{i}g}{\omega}\phi_1(x,z)\,\mathrm{e}^{-\mathrm{i}\omega t}\right], \tag{10}$$

where, assuming that the body is confined between the lines $x = \pm a$ as illustrated in Figure 1,

$$\phi_1 = \mathrm{e}^{\mathrm{i}k_1x+k_1z} + R_1\,\mathrm{e}^{-\mathrm{i}k_1x+k_1z} + \int_0^\infty B_-(\mu)(\mu\cos\mu z + k_1\sin\mu z)\,\mathrm{e}^{\mu x}\,\mathrm{d}\mu, \quad \text{in} \quad D_- \tag{11}$$

and

$$\phi_1 = T_1\,\mathrm{e}^{\mathrm{i}k_1x+k_1z} + \int_0^\infty B_+(\mu)(\mu\cos\mu z + k_1\sin\mu z)\,\mathrm{e}^{-\mu x}\,\mathrm{d}\mu, \quad \text{in} \quad D_+. \tag{12}$$

The reflection coefficient R_1, the transmission coefficient T_1 and the functions $B_\pm$ may be determined by matching ϕ_1 and its horizontal derivative on the lines $x = \pm a$ in regions $D_\pm$ to a numerical solution of the same quantities in region I.

The expansions for Φ and ζ are substituted into (6) and the right-hand side is expanded about $z = 0$ to give the free-surface boundary condition for Φ_2

$$\frac{\partial^2\Phi_2}{\partial t^2} + g\frac{\partial\Phi_2}{\partial z} = \mathrm{Re}\left[\left(-\frac{\mathrm{i}g^2}{\omega}\left(\frac{\partial\phi_1}{\partial x}\right)^2 - \frac{3\mathrm{i}\omega^3}{2}\phi_1^2 - \frac{\mathrm{i}g^2}{2\omega}\phi_1\frac{\partial^2\phi_1}{\partial x^2}\right)\mathrm{e}^{-2\mathrm{i}\omega t}\right] + \mathrm{Re}\left[\frac{\mathrm{i}g^2}{2\omega}\phi_1\frac{\partial^2\overline{\phi}_1}{\partial x^2}\right] \quad \text{on} \quad z = 0. \tag{13}$$

Thus Φ_2 may be written as

$$\Phi_2 = \mathrm{Re}[\omega\phi_{22}(x,z)\,\mathrm{e}^{-2\mathrm{i}\omega t}] + \omega\phi_{20}(x,z) \tag{14}$$

and, as has been shown in [5], the second harmonic component of Φ_2 has the far-field behaviour

$$\phi_{22} \sim \frac{\mathrm{i}R_1}{\pi}\left[\frac{\pi}{2}-\theta\right] - \frac{\mathrm{i}R_1}{4\pi k_1}\frac{x}{x^2+z^2} + \begin{cases} R_2\,\mathrm{e}^{-\mathrm{i}k_2x+k_2z}, & x\to-\infty, \\ T_2\,\mathrm{e}^{\mathrm{i}k_2x+k_2z}, & x\to\infty, \end{cases} \tag{15}$$

where $x = r\sin\theta$ and $-z = r\cos\theta$, and $k_2 = 4k_1$. The steady component of Φ_2 decays at infinity, more precisely $\phi_{20} = O(1/(x^2+z^2))$ as $x^2+z^2\to\infty$.

The free surface boundary condition for the third-order potential may be written as

$$\begin{aligned} \frac{\partial^2\Phi_3}{\partial t^2} &+ g\frac{\partial\Phi_3}{\partial z} = -2\nabla\Phi_1.\nabla\frac{\partial\Phi_2}{\partial t} - 2\nabla\Phi_2.\nabla\frac{\partial\Phi_1}{\partial t} - \frac{1}{2}\nabla\Phi_1.\nabla[(\nabla\Phi_1)^2] - \zeta_2\frac{\partial}{\partial z}\left[\frac{\partial^2\Phi_1}{\partial t^2}+g\frac{\partial\Phi_1}{\partial z}\right] \\ &- \zeta_1\frac{\partial}{\partial z}\left[\frac{\partial^2\Phi_2}{\partial t^2}+g\frac{\partial\Phi_2}{\partial z}+2\nabla\Phi_1.\nabla\frac{\partial\Phi_1}{\partial t}\right] - \frac{\zeta_1^2}{2}\frac{\partial^2}{\partial z^2}\left[\frac{\partial^2\Phi_1}{\partial t^2}+g\frac{\partial\Phi_1}{\partial z}\right] \quad \text{on} \quad z=0. \end{aligned} \tag{16}$$

The right-hand side of (16) contains terms which have frequency 3ω and terms which frequency ω, and so the third-order potential may be written as

$$\Phi_3 = \mathrm{Re}[\phi_{31}(x,z)\,\mathrm{e}^{-\mathrm{i}\omega t}] + \mathrm{Re}[\phi_{33}(x,z)\,\mathrm{e}^{-3\mathrm{i}\omega t}]. \tag{17}$$

Secular terms arise in ϕ_{31} if the right-hand side of (16) contains expressions which are of the form $\mathrm{Re}[\mathrm{e}^{\pm\mathrm{i}k_1x-\mathrm{i}\omega t}]$ or $\mathrm{Re}[(1/x)\,\mathrm{e}^{\pm\mathrm{i}k_1x-\mathrm{i}\omega t}]$, as it is straightforward to show that a solution to Laplace's equation which satistfies

$$k_1\psi - \frac{\partial\psi}{\partial z} = \mathrm{e}^{\pm\mathrm{i}k_1x} \quad \text{on} \quad z=0 \tag{18}$$

is $-(z\pm ix)\,\mathrm{e}^{\pm\mathrm{i}k_1x+k_1z}$, and a solution of Laplace's equation which satisfies

$$k_1\psi - \frac{\partial\psi}{\partial z} = \frac{\mathrm{e}^{\pm\mathrm{i}k_1x}}{x} \quad \text{on} \quad z=0 \tag{19}$$

is $(\mp\theta - \mathrm{i}\ln r)\,\mathrm{e}^{\pm\mathrm{i}k_1x+k_1z}$. Both these solutions grow as $|x|\to\infty$, for a fixed value of z. With the use of Mathematica, it may be shown that the far-field form of the free-surface boundary condition for ϕ_{31} is given by

$$k_1\phi_{31} - \frac{\partial\phi_{31}}{\partial z} \sim \begin{cases} \frac{\mathrm{i}\omega^5}{g^2}(2|R_1|^2-1)\,\mathrm{e}^{\mathrm{i}k_1x} + \frac{i\omega^3}{\pi g}|R_1|^2\frac{\mathrm{e}^{\mathrm{i}k_1x}}{x} + \frac{\mathrm{i}\omega^5}{g^2}R_1(2-|R_1|^2)\,\mathrm{e}^{-\mathrm{i}k_1x} + \frac{i\omega^3}{\pi g}R_1\frac{\mathrm{e}^{-\mathrm{i}k_1x}}{x} & \\ + \quad \text{non-secular terms,} & x\to-\infty, \\ -\frac{\mathrm{i}\omega^5}{g^2}T_1|T_1|^2\,\mathrm{e}^{\mathrm{i}k_1x} + \frac{i\omega^3}{\pi g}R_1\overline{T}_1\frac{\mathrm{e}^{-\mathrm{i}k_1x}}{x} + \quad \text{non-secular terms,} & x\to\infty. \end{cases} \tag{20}$$

The terms which give rise to secularity in ϕ_{31} have come from self- and cross- interactions between the first-order incident, reflected and transmitted waves and interaction between the first-order waves and the far-field form of ϕ_{22}. In the next section the secular terms are eliminated by a modification to the form of the first-order waves.

A modified expansion for Φ

The first-order potential is modified in regions $D_\pm$ so that it is given by

$$\phi_1 = \mathrm{e}^{f_i(x,z,A)} + R_1\,\mathrm{e}^{f_r(x,z,A)} + \int_0^\infty B_-(\mu)(\mu\cos\mu z + k_1\sin\mu z)\,\mathrm{e}^{\mu x}\,\mathrm{d}\mu, \quad \text{in} \quad D_- \tag{21}$$

and

$$\phi_1 = T_1\,\mathrm{e}^{f_{t1}(x,z,A)} - \mathrm{i}\sin(f_{t2}(x,z,A))\,\mathrm{e}^{-\mathrm{i}k_1x+k_1z} + \int_0^\infty B_+(\mu)(\mu\cos\mu z + k_1\sin\mu z)\,\mathrm{e}^{-\mu x}\,\mathrm{d}\mu, \quad \text{in} \quad D_+, \tag{22}$$

where

$$f_i(x,z,A) = (k_1 + A^2 k_{i2})[\mathrm{i}(x + A^2\alpha_i \ln r) + (z + A^2\alpha_i(\theta + \pi/2))] + O((k_1A)^3), \tag{23}$$

$$f_r(x,z,A) = (k_1 + A^2 k_{r2})[\mathrm{i}(x + A^2\alpha_r \ln r) + (z + A^2\alpha_r(\theta + \pi/2))] + O((k_1A)^3), \tag{24}$$

$$f_{t1}(x,z,A) = (k_1 + A^2 k_{t2})(\mathrm{i}x + z) + O((k_1A)^3), \tag{25}$$

$$f_{t2}(x,z,A) = \alpha_t k_1 A^2(\ln r + \mathrm{i}(\theta - \pi/2)) + O((k_1A)^3) \tag{26}$$

and k_{i2}, k_{r2}, k_{t2}, α_{i2}, α_{r2} and α_{t2} are constants to be determined. It is straightforward to show that this modified form of the potential still satisfies Laplace's equation in regions $D_{\pm}$, but if ϕ_1 is unaltered in region I then there are jumps in the potential $\Phi - A\Phi_1$ and its horizontal derivative on $x = \pm a$ at $O((k_1A)^3)$. If the modified form of the first-order potential is substituted into the free surface boundary condtion (6) and this is expanded to $O((k_1A)^3)$, and the $O((k_1A)^2)$ terms neglected in the exponents at every order, then with the use of Mathematica it is straightforward to show that the secular terms in ϕ_{31} may be eliminated by choosing the constants to be

$$k_{i2} = \frac{\omega^6}{g^2}(2|R_1|^2 - 1), \quad k_{r2} = \frac{\omega^6}{g^2}(2 - |R_1|^2), \quad k_{t2} = -\frac{\omega^6}{g^2}|T_1|^2 \tag{27}$$

and

$$\alpha_i = \frac{\omega^2|R_1|^2}{\pi g}, \quad \alpha_r = \frac{\omega^2}{\pi g} \quad \text{and} \quad \alpha_t = \frac{\omega^2 R_1 \overline{T}_1}{\pi g}. \tag{28}$$

Discussion

The nonlinear modifications to the waves determined by the constants k_{i2}, k_{r2} and k_{t2} defined in (27) are precisely of the type found by Stokes ([6]), although it is interesting to note that the nonlinear interactions mean that the constants are not the same as they would be if each wave were considered in isolation. However the form of the modified expansion for ϕ_1 given in (21) and (22) can be used to satisfy the boundary value problems at each order only if the $O((k_1A)^2)$ terms in the exponents are neglected at every order. This is not totally satisfactory because there is no guarantee that these terms form the leading order terms to an exact nonlinear potential, and it is hoped to report further progress on this issue at the workshop.

References

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