Scattering by a semi-infinite array of vertical circular cylinders

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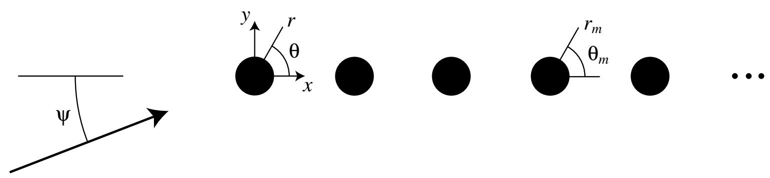
SUMMARY

Scattering by large finite arrays is of considerable importance in the context of the design of offshore structures supported by thousands of cylindrical columns. It has long been recognised that one way to approach large finite array scattering is to analyse the effects of each edge of the array in isolation—in other words to study arrays with just one edge—and this leads to problems formulated on semi-infinite arrays. Unfortunately, such problems are difficult to analyse and little work has been done on the subject since the pioneering studies of Hills & Karp [1] and Millar [2]. As a first step we consider here the case of a semi-infinite row of periodically-spaced, vertical, circular cylinders extending throughout the water depth and show how the diffracted field can be efficiently computed.

FORMULATION

We consider water of depth h and under the usual assumptions of linear water wave theory, look for solutions of Laplace's equation of the form $\Phi(x, y, z, t) = \mathrm{Re}[\phi(x, y) \cosh(k(z + h)) \exp(-\mathrm{i}\omega t)]$, where k is the positive solution to the dispersion relation $k \tanh kh = \omega^2/g$, ω is the angular frequency, and g is the acceleration due to gravity. The geometry under consideration is sketched in plan view below:



We are concerned with the scattering of a plane wave

$$\phi_{\mathrm{inc}} = \mathrm{e}^{\mathrm{i}(\beta x + \alpha y)},$$

where $\alpha = k \sin\psi$ and $\beta = k \cos\psi$, by a semi-infinite row of identical rigid circular cylinders of radius a, located at $(x, y) = (ms, 0)$, $m = 0, 1, 2, \ldots$, where s is the spacing. We will use polar coordinates (r_m, θ_m), $\theta_m \in (-\pi, \pi]$, centred on the mth scatterer and defined by

$$x - ms = r_m \cos\theta_m, \qquad y = r_m \sin\theta_m$$

and we will usually write (r, θ) for (r_0, θ_0). In terms of (r_m, θ_m) the incident wave is

$$\phi_{\mathrm{inc}} = I_m \mathrm{e}^{\mathrm{i}kr_m \cos(\theta_m - \psi)},$$

where

$$I_m = \mathrm{e}^{\mathrm{i}\beta ms}.$$

It is not difficult to formulate this problem using separation of variables. If we write the total field as

$$\phi = \phi_{\mathrm{inc}} + \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} A_n^m Z_n H_n^{(1)}(kr_m) \mathrm{e}^{\mathrm{i}n\theta_m}, \tag{1}$$

where $Z_n = J_n'(ka)/H_n^{(1)\prime}(ka)$, then the unknowns A_n^m are solutions to ([3, eqn 2.11])

$$A_m^p + \sum_{n=-\infty}^{\infty} Z_n \left(\sum_{j=0}^{p-1} A_n^j H_{n-m}^{(1)}(k|j-p|s) + \sum_{j=p+1}^{\infty} (-1)^{n-m} A_n^j H_{n-m}^{(1)}(k|j-p|s) \right) = -I_p \mathrm{e}^{\mathrm{i}m(\frac{1}{2}\pi - \psi)}, \tag{2}$$

$p = 0, 1, 2, \ldots$, $m = 0, \pm 1, \pm 2, \ldots$. This system of equations could, in principle, be solved numerically by truncation but the infinite sum from $j = p + 1$ to infinity converges far too slowly for this to be practicable. (The terms in this sum decay like $j^{-1/2} \exp(\mathrm{i}j\theta)$ for some θ.) Instead, we solve for the difference between the coefficients A_m^p and those that arise in the much simpler infinite array problem in which the cylinder array extends to both plus and minus infinity.

Infinite grating

For the infinite grating problem, with cylinders at $(ms, 0)$, $m = 0, \pm 1, \pm 2, \ldots$, we seek a solution of the form

$$\phi = \phi_{\text{inc}} + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} B_n^m Z_n H_n^{(1)}(kr_m) \mathrm{e}^{\mathrm{i}n\theta_m}. \tag{3}$$

The periodicity of the geometry and of the incident wave implies that $B_n^m = I_m B_n^0 = I_m B_n$, say, and we only need to solve for B_n. These coefficients are solutions to the infinite system of equations

$$B_m + \sum_{n=-\infty}^{\infty} B_n Z_n \sigma_{n-m} = -\mathrm{e}^{\mathrm{i}m(\frac{1}{2}\pi - \psi)}, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{4}$$

where

$$\sigma_n = \sum_{j=1}^{\infty} \left[(-1)^n I_j + I_{-j} \right] H_n^{(1)}(kjs).$$

The quantities σ_n are easily evaluated (though not from the above expression). For $n = 0$ we can use the formulas given in [4], whereas for $n > 0$ suitable expressions can be found in [5].

The far field can be determined as follows. If we insert the integral representation

$$H_n^{(1)}(kr)\mathrm{e}^{\mathrm{i}n\theta} = \frac{(-\mathrm{i})^{n+1}}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-k\gamma(t)|y|}}{\gamma(t)} \, \mathrm{e}^{\mathrm{i}kxt} \, \mathrm{e}^{\mathrm{i}n \operatorname{sgn}(y) \arccos t} \, \mathrm{d}t, \tag{5}$$

in which

$$\gamma(t) = \sqrt{t^2 - 1} = -\mathrm{i}\sqrt{1 - t^2}$$

and care must be taken over the choice of branch for the arccos function, into (3) and then apply the Poisson summation formula we get

$$u = u_{\text{inc}} + \sum_{n=-\infty}^{\infty} (-\mathrm{i})^n B_n Z_n \sum_{m=-\infty}^{\infty} \frac{2\mathrm{e}^{\mathrm{i}kr\cos(\theta - \operatorname{sgn}(\theta)\psi_m)}}{ks \sin\psi_m} \, \mathrm{e}^{\mathrm{i}n \operatorname{sgn}(\theta)\psi_m}.$$

Here we have defined the scattering angles ψ_m by

$$\psi_m = \arccos(\beta_m / k), \qquad \beta_m = \beta + 2m\pi/s.$$

The only terms which contribute to the far field are those for which $|\beta_m| < k$, i.e.

$$-1 < \cos\psi + \frac{2m\pi}{ks} < 1.$$

We then say that $m \in \mathcal{M}$ and we have $0 < \psi_m < \pi$. Thus, as $y \to \pm\infty$, the far field consists of a set of plane waves propagating in the directions $\theta = \pm\psi_m$:

$$u \sim u_{\text{inc}} + \sum_{m \in \mathcal{M}} \frac{2\mathrm{e}^{\mathrm{i}kr\cos(\theta \mp \psi_m)}}{ks \sin\psi_m} \sum_{n=-\infty}^{\infty} (-\mathrm{i})^n B_n Z_n \, \mathrm{e}^{\pm \mathrm{i}n\psi_m}.$$

Semi-infinite grating

We define a new set of unknowns, C_m^p, corresponding to the difference between the infinite and semi-infinite solutions:

$$A_m^p = C_m^p + I_p B_m. \tag{6}$$

We expect that as $p \to \infty$ (i.e. as we move away from the edge) the coefficients A_m^p will tend to the values appropriate to a fully infinite array, and hence that $C_m^p \to 0$ as $p \to \infty$. If we substitute from (6) into (2) and use (4) we get a system of equations for the coefficients C_m^p which is the same as (2) except that now the right-hand side is

$$\sum_{n=-\infty}^{\infty} B_n Z_n \sum_{j=p+1}^{\infty} I_{p-j} H_{n-m}^{(1)}(kjs).$$

The sum over j is the slowly-convergent series that causes all the problems, but now it does not contain unknown coefficients and so can be treated analytically. In fact, we have found that

$$\sum_{j=1}^{\infty} I_{-j} H_n^{(1)}(kjs) = (-1)^{n+1}\frac{\mathrm{i}}{\pi}\int_0^{\infty} \frac{\mathrm{e}^{\mathrm{i}n\arccos t} + (-1)^n \mathrm{e}^{-\mathrm{i}n\arccos t}}{\gamma(\mathrm{e}^{\mathrm{i}\beta s}\,\mathrm{e}^{ks\gamma} - 1)}\,\mathrm{d}t + \sum_{0<\beta_m<k} \frac{2(-\mathrm{i})^n \cos n\psi_m}{ks\sin\psi_m}$$

from which all the necessary sums can be efficiently evaluated.

From (1) and (5) the total field is

$$\phi = \phi_{\mathrm{inc}} + \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} (C_n^m + I_m B_n) Z_n \frac{(-\mathrm{i})^{n+1}}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-k\gamma(t)|y|}}{\gamma(t)}\,\mathrm{e}^{\mathrm{i}k(x-ms)t}\,\mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(y) n\arccos t}\,\mathrm{d}t.$$

The determination of the asymptotic form of this expression for large kr is quite involved. We use the formula

$$\sum_{m=0}^{\infty}\int_{-\infty}^{\infty} f(u)\,\mathrm{e}^{-\mathrm{i}mu}\,\mathrm{d}u = \int_{-\infty}^{\infty} \frac{f(u)}{1-\mathrm{e}^{-\mathrm{i}u}}\,\mathrm{d}u,$$

which can be derived using generalized functions, to express the sum over m involving I_m as a contour integral. Stationary phase can then be used to give the far-field form of the potential. We find that

$$\phi \sim \tilde{H}(kr) g(\theta) + \sum_{\substack{m\in\mathcal{M}\\ \psi_m>|\theta|}} \frac{2\,\mathrm{e}^{\mathrm{i}kr\cos(|\theta|-\psi_m)}}{ks\sin\psi_m}\sum_{n=-\infty}^{\infty} (-\mathrm{i})^n B_n Z_n \mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(\theta) n\psi_m}, \tag{7}$$

where $\tilde{H}(kr) = \sqrt{2/\pi kr}\,\exp(\mathrm{i}(kr - \frac{1}{4}\pi))$ and

$$g(\theta) = \sum_{n=-\infty}^{\infty} (-\mathrm{i})^n \mathrm{e}^{\mathrm{i}n\theta} Z_n \left(\frac{B_n}{1-\mathrm{e}^{\mathrm{i}ks(\cos\psi-\cos\theta)}} + \sum_{m=0}^{\infty} C_n^m\,\mathrm{e}^{-\mathrm{i}kms\cos\theta} \right).$$

A number of things are worthy of note. Firstly, the diffracted field takes the form of a circular wave of 'amplitude' $g(\theta)$ plus a sum of plane waves which propagate in the same directions as for the infinite grating case. However, unlike in the grating problem, the plane waves do not exist everywhere and the wave making an angle ψ_m (resp. $-\psi_m$) with the x-axis is only found in the sector $0 < \theta < \psi_m$ (resp. $0 > \theta > -\psi_m$). Secondly, the coefficients C_n^m affect only the circular wave. The plane wave field is determined entirely from the solution to the infinite grating problem; in fact, where the plane waves exist, their amplitude is precisely as in the infinite grating problem. Thus it is only the circular wave which causes any computational difficulties. Thirdly, it is apparent that the amplitude $g(\theta)$ of the circular wave becomes infinite as θ approaches one of the so-called shadow boundaries $|\theta| = \psi_p$. This is because, in performing the steepest descent analysis we have explicitly assumed that $|\theta| \neq \psi_p$, a case which corresponds to a pole of the integrand coinciding with the appropriate saddle point. Uniform asymptotics valid as $\theta \to \psi_p$ can be derived. A lengthy calculation reveals that near these lines we must add a term

$$\tilde{g}(r,\theta) = \sum_{n=-\infty}^{\infty} \frac{B_n Z_n (-\mathrm{i})^{n+1} \mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(\theta) n\psi_p}}{2ks\sin\frac{1}{2}(|\theta|-\psi_p)\sin\psi_p}\left(1 + 2\mathrm{i}\zeta_p\,\mathrm{e}^{-\mathrm{i}\zeta_p^2} F(\zeta_p)\right),$$

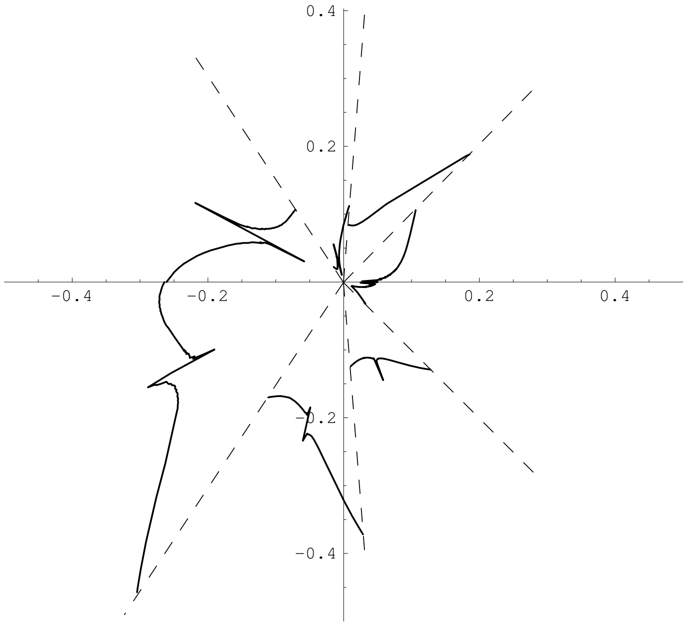
where $\zeta_p = \sqrt{2kr}\sin\frac{1}{2}||\theta| - \psi_p|$ and F is the Fresnel integral defined by

$$F(v) = \int_v^{\infty} \mathrm{e}^{\mathrm{i}u^2}\,\mathrm{d}u \qquad (0 < \arg u < \tfrac{1}{2}\pi \text{ as } u \to \infty).$$

The combination $g + \tilde{g}$ is bounded as $\theta \to \psi_p$ for any r, but the limit is different from each side. However, since $F(0) = \frac{1}{2}\sqrt{\pi}\exp(\mathrm{i}\pi/4)$, the discontinuity in $g + \tilde{g}$ as θ passes through ψ_p exactly cancels the extra plane wave contribution that appears in (7) as the shadow boundary is crossed, thus ensuring that the far field is continuous.

RESULTS AND DISCUSSION

Typical results that can be obtained are shown below. The figure shows the amplitude of the circular wave, $|g(\theta)+\tilde{g}(r,\theta)|$, for the case $ka=0.05$, $ks=10$, $kr=20$, and $\psi=\pi/4$. There are three predominant scattering directions corresponding to $\pm\arccos(1/\sqrt{2}-m\pi/5)$, $m=0,1,2$, i.e. $\pm\pi/4$, $\pm0.475\pi$, $\pm0.685\pi$ and these are indicated by the dashed lines.



The discontinuities in $|g+\tilde{g}|$ are clearly apparent, as are a number of other sharp features. In the related mathematical problem where we impose $\phi=0$ on the boundaries of the cylinders and assume that $ka \ll 1$ it is possible to show analytically (see [6]) that the amplitude of the circular wave is zero in the directions which correspond to the scattering angles when the incident wave angle is zero. In this case these are $\pm\arccos(1-m\pi/5)$, $m=0,1,2,3$, i.e. 0, $\pm0.379\pi$, $\pm0.583\pi$, and $\pm0.846\pi$. Reference to the figure shows that these are precisely the angles at which the spiky behaviour is found, though as yet no analytical reason for this has been found.

We have assumed throughout that none of the scattering angles are 0 or π. The first case is usually termed inward resonance and the latter outward resonance. In these cases many of the series that have been derived above do not converge and a separate analysis must be undertaken. The inward resonance case turns out to be rather involved, but the scattered field for outward resonance can be easily determined.

References

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