Water Wave Scattering By The Two Edges Of A Strip-like Ice-cover

by

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Summary : Scattering of surface water waves by the two edges of a strip-like ice-cover, modelled as a thin elastic plate, floating on clean water is studied for the case of deep water. The main problem is reduced to solving two coupled Carleman-type singular integral equations which are solved approximately by assuming the breadth of the strip to be large. The numerical results for the reflection coefficient are depicted graphically against the ice-cover parameter and also against the wavenumber. Oscillatory nature of the reflection coefficient is found to be the prime feature of the graphs.

1. Introduction : The main interest in the investigation of scattering problems involving ice-sheet modelled as an elastic plate lies in the fact that these problems have wide applications in designing floating airports, offshore pleasure cities etc. Chakrabarti (2000) obtained analytically the reflection and transmission coefficients of the scattering problem involving a semi-infinite ice-cover, by solving a Carleman-type singular integral equation by the technique of Riemann-Hilbert problem. Linton and Chung (2003) used residue calculus technique in the mathematical analysis to investigate this problem. Tkacheva (2003)considered the ice-wave interaction problem wherein the ice-cover is a strip of finite width floating on water of finite depth. She investigated the problem by reducing it to a three-part Wiener-Hopf technique. In the present paper the ice-cover is taken in the form of a finite strip and the corresponding scattering problem is solved by using Havelock's expansion followed by reducing it to two coupled Carleman-type singular integral equations, which are solved approximately after assuming the breadth of the strip to be large.

2. Formulation : For the mathematical analysis of the problem we assume linear theory and irrotational motion and take a two-dimensional Cartesian coordinates (x, y). Suppose water occupies the region $y > 0, -\infty < x < \infty$ and a part of the water surface $0 < x < l, y = 0$ is covered by a thin sheet of ice. If $Re\{\phi(x, y)e^{-i\sigma t}\}$ denotes the governing velocity potential then ϕ satisfies

$$\phi_{xx} + \phi_{yy} = 0, -\infty < x < \infty,\ y > 0, \tag{2.1}$$

$$K\phi + \phi_y = 0 \text{ on } y = 0, -\infty < x < 0, l < x < \infty, \tag{2.2}$$

where $K = \frac{\sigma^2}{g}$, g being the gravity,

$$K\phi + \phi_y + D\phi_{yxxxx} = 0 \text{ on } y = 0, 0 < x < l, \tag{2.3}$$

where $D = \frac{Eh_0^3}{12(1-\nu^2)\rho g}$ is the flexural rigidity of the material of the ice-cover,

$$\nabla\phi \to 0 \text{ as } y \to \infty \tag{2.4}$$

$$\phi(x, y) \sim \begin{cases} e^{-Ky+iKx} + Re^{-Ky-iKx} & \text{as } x \to -\infty, \\ Te^{-Ky+iK(x-l)} & x \to \infty, \end{cases} \tag{2.5}$$

R and T are the unknown reflection and transmission coefficients respectively to be determined,

$$\phi_{yxx} \to 0 \text{ as } x \to 0+0, l-0, y = 0 \tag{2.6}$$

$$\phi_{yxxxx} \to 0 \text{ as } x \to 0+0, l-0, y = 0 \tag{2.7}$$

Also

$$\phi(x, y) = \begin{cases} \alpha e^{-\lambda Ky+i\lambda Kx} + \beta e^{-\lambda Ky-i\lambda K(x-l)} \\ +A_1e^{-\lambda_1 Ky+i\lambda_1 Kx} + A_2e^{-\lambda_1 Ky-i\lambda_1 Kx} \\ +A_3e^{-\bar{\lambda}_1 Ky+i\bar{\lambda}_1 Kx} + A_4e^{-\bar{\lambda}_1 Ky-i\bar{\lambda}_1 Kx} \\ +\chi(x, y) \end{cases} \tag{2.8}$$

where $\alpha, \beta, A_i(i = 1, ..., 4)$ are unknown constants and $\chi(x, y)$ is an unknown and non-propagating solution of Laplace equation. λK and $(\lambda_1 K, \bar{\lambda}_1 K)$ are respectively the real and complex roots of the equation

$$Dz^5 + z - K = 0 \tag{2.9}$$

whose another pair of complex roots are $(\lambda_2 K, \bar{\lambda}_2 K)$. Here $Re(\lambda_1 K) > 0$, $Re(\lambda_2 K) < 0$, $Im(\lambda_1 K, \lambda_2 K) > 0$.

3. Reduction to Carleman-type singular integral equation: Let $\phi(x,y)=\psi_{xx}(x,y)$, then $\psi(x,y)$ has the following representations:

$$\psi(x,y)=-\frac{1}{K^2}e^{-Ky+iKx}-\frac{R}{K^2}e^{-Ky-iKx}+\frac{2}{\pi}\int_0^\infty \frac{A(\xi)L(\xi,y)e^{\xi x}}{\xi^2+K^2}d\xi,\ x<0, \tag{3.1}$$

$$\begin{aligned}\psi(x,y)=&-\frac{1}{\lambda^2K^2}\{\alpha e^{-\lambda Ky+i\lambda Kx}+\beta e^{-\lambda Ky-i\lambda K(x-l)}\}\\&-\frac{1}{\lambda_1^2K^2}\{A_1e^{-\lambda_1Ky+i\lambda_1Kx}+A_2e^{-\lambda_1Ky-i\lambda_1Kx}\}\\&-\frac{1}{\bar{\lambda_1}^2K^2}\{A_3e^{-\bar{\lambda}_1Ky+i\bar{\lambda}_1Kx}+A_4e^{-\bar{\lambda}_1Ky-i\bar{\lambda}_1Kx}\}\\&+\frac{2}{\pi}\int_0^\infty\frac{B(\xi)e^{\xi x}+C(\xi)e^{-\xi x}}{\xi^2(D\xi^4+1)^2+K^2}M(\xi,y)d\xi,0<x<l,\end{aligned} \tag{3.2}$$

$$\psi(x,y)=-\frac{T}{K^2}e^{-Ky+iK(x-l)}+\frac{2}{\pi}\int_0^\infty\frac{S(\xi)L(\xi,y)}{\xi^2+K^2}d\xi,x>l, \tag{3.3}$$

where

$$L(\xi,y)=\xi\cos\xi y-K\sin\xi y$$

and

$$M(\xi,y)=\xi(D\xi^4+1)\cos\xi y-K\sin\xi y,$$

$A(\xi),B(\xi),C(\xi)$ and $S(\xi)$ are unknown functions to be determined. Use of continuity of ψ and ψ_x across the lines $x=0$ and $x=l$ and then application of Havelock's expansion theorem (Ursell (1947)) will produce the following relations between the unknown constants α,β,R,T and the unknown functions $B(\xi)$ and $C(\xi)$.

$$\begin{aligned}-\frac{1}{2K^3}-\frac{R}{2K^3}=&-\frac{\alpha}{K^3\lambda^2(\lambda+1)}-\frac{\beta e^{i\lambda Kl}}{K^3\lambda^2(\lambda+1)}\\&-\frac{A_1+A_2}{K^3\lambda_1^2(\lambda_1+1)}-\frac{A_3+A_4}{K^3\bar{\lambda_1}^2(\bar{\lambda_1}+1)}\\&+\frac{2}{\pi}\int_0^\infty\frac{\{B(\xi)+C(\xi)\}DK\xi^5}{(\xi^2+K^2)\{\xi^2(D\xi^4+1)^2+K^2\}}d\xi\end{aligned} \tag{3.4}$$

$$\begin{aligned}-\frac{i}{2K^2}+\frac{iR}{2K^2}=&-\frac{i\alpha}{K^2\lambda(\lambda+1)}+\frac{i\beta e^{i\lambda Kl}}{K^2\lambda(\lambda+1)}\\&-\frac{i(A_1-A_2)}{K^2\lambda_1(\lambda_1+1)}-\frac{i(A_3-A_4)}{K^2\bar{\lambda_1}(\bar{\lambda_1}+1)}\\&+\frac{2}{\pi}\int_0^\infty\frac{\{B(\xi)-C(\xi)\}DK\xi^6}{(\xi^2+K^2)\{\xi^2(D\xi^4+1)^2+K^2\}}d\xi\end{aligned} \tag{3.5}$$

$$\begin{aligned}-\frac{T}{2K^3}=&-\frac{\alpha e^{i\lambda Kl}}{K^3\lambda(\lambda+1)}-\frac{\beta}{K^3\lambda(\lambda+1)}\\&-\frac{(A_1e^{i\lambda_1Kl}+A_2e^{-i\lambda_1Kl})}{K^3\lambda_1^2(\lambda_1+1)}-\frac{(A_3e^{i\bar{\lambda}_1Kl}+A_4e^{-i\bar{\lambda}_1Kl})}{K^3\bar{\lambda_1}^2(\bar{\lambda_1}+1)}\\&+\frac{2}{\pi}\int_0^\infty\frac{\{B(\xi)e^{\xi l}+C(\xi)e^{-\xi l}\}DK\xi^5}{(\xi^2+K^2)\{\xi^2(D\xi^4+1)^2+K^2\}}d\xi\end{aligned} \tag{3.6}$$

$$\begin{aligned}-\frac{iT}{2K^2}=&-\frac{i\alpha e^{i\lambda Kl}}{K^2\lambda(\lambda+1)}+\frac{i\beta}{K^2\lambda(\lambda+1)}\\&-\frac{i(A_1e^{i\lambda_1Kl}-A_2e^{-i\lambda_1Kl})}{K^2\lambda_1^2(\lambda_1+1)}-\frac{i(A_3e^{i\bar{\lambda}_1Kl}-A_4e^{-i\bar{\lambda}_1Kl})}{K^3\bar{\lambda_1}^2(\bar{\lambda_1}+1)}\\&+\frac{2}{\pi}\int_0^\infty\frac{\{B(\xi)e^{\xi l}-C(\xi)e^{-\xi l}\}DK\xi^6}{(\xi^2+K^2)\{\xi^2(D\xi^4+1)^2+K^2\}}d\xi\end{aligned} \tag{3.7}$$

Also this gives rise to the following coupled Carleman-type singular integral equations:

$$\lambda(\xi)B_1(\xi)+\frac{1}{\pi}\int_0^\infty\frac{B_1(u)}{u-\xi}du-\frac{1}{\pi}\int_0^\infty\frac{C_1(u)e^{-ul}}{u+\xi}du=F_B(\xi),\ \xi>0, \tag{3.8}$$

$$\lambda(\xi)C_1(\xi)+\frac{1}{\pi}\int_0^\infty\frac{C_1(u)}{u-\xi}du-\frac{1}{\pi}\int_0^\infty\frac{B_1(u)e^{-ul}}{u+\xi}du=F_C(\xi),\ \xi>0, \tag{3.9}$$

where

$$B_1(\xi),\ C_1(\xi)=\frac{DK\xi^5}{\xi^2(D\xi^4+1)^2+K^2}\{B(\xi)e^{\xi l},C(\xi)\},$$

$$\lambda(\xi)=\frac{\xi^2(D\xi^4+1)^2+K^2}{DK\xi^5},$$

$F_B(\xi)$ and $F_C(\xi)$ are functions involving the constants.

As $l \to \infty$ the above equations reduce to the following uncoupled equations:

$$\lambda(\xi)B_1^o(\xi)+\frac{1}{\pi}\int_0^\infty \frac{B_1^o(u)}{u-\xi}du = F_B^o(\xi),\ \xi>0, \quad (3.10)$$

$$\lambda(\xi)C_1^o(\xi)+\frac{1}{\pi}\int_0^\infty \frac{C_1^o(u)}{u-\xi}du = F_C^o(\xi),\ \xi>0, \quad (3.11)$$

where the superscript 'o' denotes the zero-order approximation to the functions $B_1(\xi)$ and $C_1(\xi)$ and various constants appearing in $F_B^0(\xi)$ and $F_C^0(\xi)$.

Assuming the right-hand sides of the integral equations (3.10) and (3.11) to be known these are solved by reducing to Riemann-Hilbert problems. This will produce $B_1^0(\xi)$ and $C_1^0(\xi)$ in terms of the unknowns $\alpha^0, \beta^0, R^0, T^0, A_1^0, A_2^0, A_3^0, A_4^0$. Substituting $B_1^0(\xi)$ and $C_1^0(\xi)$ into the relations (3.4) to (3.7) we will get four relations for the determination of the eight unknown constants. The other four relations will be obtained from conditions (2.6) and (2.7). Solving the linear system of eight equations in eight unknowns we will get $\alpha^0, \beta^0, R^0, T^0, A_1^0, A_2^0, A_3^0, A_4^0$.

To get the higher-order approximations we put $B_1^0(\xi)$, $C_1^0(\xi)$ into (3.8) and (3.9) and obtain

$$\lambda(\xi)B_1^1(\xi)+\frac{1}{\pi}\int_0^\infty \frac{B_1^1(u)}{u-\xi}du = G_B^1(\xi),\ \xi>0, \quad (3.12)$$

$$\lambda(\xi)C_1^1(\xi)+\frac{1}{\pi}\int_0^\infty \frac{C_1^1(u)}{u-\xi}du = G_C^1(\xi),\ \xi>0, \quad (3.13)$$

where

$$G_{B,C}^1(\xi) = \frac{1}{\pi}\int_0^\infty \frac{[C_1^0(u), B_1^0(u)]e^{-ul}}{u+\xi}du + F_{B,C}^1(\xi).$$

Here the superscript '1' denotes the first-order approximation to the functions $B_1(\xi), C_1(\xi)$ and the unknown constants involved in $F_{B,C}^1(\xi)$.

Proceeding as before, the unknown functions $B_1^1(\xi)$ and $C_1^1(\xi)$ can be obtained by a similar technique. Once $B_1^1(\xi)$ and $C_1^1(\xi)$ are determined the first-order approximations to the unknown constants $R, T, \alpha, \beta, A_1, A_2, A_3, A_4$ are computed numerically by solving the eight equations comprising of the four relations (3.4) to (3.7) and another four relations obtained by using conditions (2.6) and (2.7). This procedure can be repeated to get more higher-order approximations. However this is not pursued here as the first-order approximations give sufficiently accurate results.

4. Numerical Results : For numerical calculation we have taken a characteristic length L in order to nondimensionalize different parameters. We have plotted the reflection coefficient $|R|$ against the ice-cover parameter $\frac{D}{L^4}$ for $KL = 1$ in the Figure 1, taking the strip-breadth $l/L = 10$. $|R|$ exhibits oscillatory nature against $\frac{D}{L^4}$, which may be attributed due to multiple reflections of the incident wave field by the two edges of the strip of the ice-cover. Figures for $|R|$ against the wave-number KL for fixed values of $\frac{D}{L^4}$ and l/L, have also been drawn, and these exhibit highly oscillatory nature. However, these figures are not shown here.

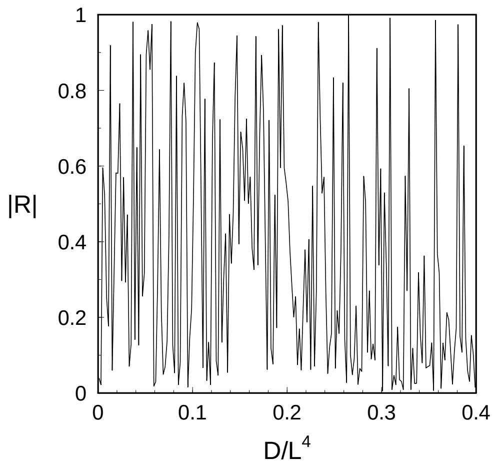

Figure1 : Reflection coefficient for KL=1 and l/L=10

5. Conclusion : The boundary value problem arising in scattering of surface water waves by an ice-strip modelled as a thin elastic plate is studied by reducing the problem to solving a pair of coupled Carleman-type singular integral equations. Graphs for the reflection coefficient show its highly oscillatory nature which is due to multiple reflection by the two edges of the strip. The method can be employed to study scattering problems involving two or more finite ice-strips and also to the case when a finite strip of free surface is embedded between two semi-infinite ice-strips.

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Discusser: M.P. Tulin

The reflection of a wave requires that a horizontal force be exerted on the wave system. For a flat plate it would seem that this force must originate at the edge of the plate. How can this be in the case of an infinitely thin horizontal plate?

It would seem to follow that the actual thickness of the plate must appear as a parameter in the scattering problem. From the mathematical point of view, the limiting process must involve both the wave amplitude, a, and the plate thickness, δ. Then, even though both a and $\delta \to 0$, the ratio (a/δ) must appear as a parameter. Physically, if $\delta/a \ll 1$, the tops of the waves well flow onto the top of the plate, whereas, if $\delta/a \gg 1$ water will collect in the front of the edge, causing breaking.

Author's reply:

Our problem may be considered as a model for a large floating structure in which the thickness of the plate is small compared to the depth of the water not to the Wave amplitude. Though we have not done any experimental study, we are sure from the analysis and numerical results that some parts of the incoming wave will be reflected by the two edges of ice-strip, whatever be the wave amplitude.

Because as a whole the ice-strip is very large in magnitude. It is infinitely long and also, according to our problem, the width of the strip is large. If the thickness is infinitely small, then only a prime part of the incident wave will be transmitted as I showed in the last figure.

Discusser: D. Porter

1. Does iterative process converge?
2. Why is only first correction used?

Author's reply:

1. Yes, the iteration process converges.
2. As we could not find any paper in which the numerical results for $|R|$ for an ice-strip in 'deep-water' are given, we look the results for $|R|$ up to first order based on the following two facts. (i) The graphs of $|R'|$ show relevant features for different physical situation such as when D is small $|R'|$ is small etc.. (ii) Few months ago we studied a problem involving two different initial surfaces floating on the free surface. From that problem we approached in a similar manner. We found that the graphs of $|R'|$ exactly coincide with those obtained earlier in Kanoria *et al.* (Wave motion, 1999) who got the results by reducing the problem to a three-part Weiner-Hopf problem. Thus, in the present problem also we have not pursued beyond the first order approximation.