NEW ASYMPTOTIC FORMULATION FOR NONLINEAR WATER WAVE PROBLEMS IN LAGRANGIAN COORDINATES

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SUMMARY

A new asymptotic description of water wave motion in Lagrangian coordinates is developed. The method is applied to problems of 2D regular travelling waves in deep water and standing Faraday waves. The fifth order asymptotic solution for travelling waves and the third order solution for standing waves are obtained. The travelling wave solution is uniformly valid at all times. The free surface predictions of both solutions are considerably better then the predictions of the Eulerian asymptotic methods of the same order. The approach described can in future be applied to a wide range of strongly nonlinear water wave problems, including diffraction, fluid-structure interaction, and so on.

1 INTRODUCTION

The main complication of problems with a free surface is that the Eulerian position of this surface is not specified *a priori*, and must be found as a part of the solution. The Lagrangian approach, when the motion of individual fluid particles is considered, is the natural way to overcome this difficulty. For a wide class of flows the free surface remains a single connected domain and is free of singularities. In this case surface particles stay on the surface throughout the entire motion. This means that in the Lagrangian description the free surface is represented by a fixed boundary of the domain in the space of Lagrangian indexes. The limitation on this is for certain complex surface phenomena, where self-contact by different parts of the free surface occurs.

The equations describing the motion of water waves are nonlinear and complicated for both theoretical analysis and numerical modelling. Various types of asymptotic techniques can be used to simplify the problem formulation to such an extent that it can be efficiently treated by analytical or numerical methods. The first successful application of an asymptotic method to a water waves problem was the Stokes expansion for a regular travelling wave. From a view point of modern asymptotic methods this solution is a regular asymptotic expansion with respect to a small wave steepness. The Stokes expansion provides a basis of perturbation techniques for problems of significant practical interest, such as wave diffraction, floating body behaviour, and so on. Due to the complexity of the formulation, however, only linear or low-order nonlinear perturbation theories are usually used for engineering applications. Unfortunately, certain important features of wave motion cannot be captured by classical low-order asymptotic expansions.

Nonlinearity enters Eulerian and Lagrangian equations of fluid motion in different ways. Moreover, the Lagrangian description is more appropriate for the description of violent free surface motion. Thus, we could expect that the perturbation technique applied to the equations in the Lagrangian form would be able to describe effects that cannot be captured by classical Eulerian expansions. For example, such phenomena as overturning waves can in principle be described by low order perturbation methods if applied to equations written in Lagrangian coordinates. There have been limited attempts to follow this approach. For example, Pierson (1962) made an attempt to apply perturbation expansions to water wave problems with the Lagrangian formulation. He represented the Lagrangian displacement of fluid particles as a regular expansion with respect to powers of a small parameter. Application of this approach to the problem of a regular travelling wave leads to a solution which is non-uniformly valid for large times when, due to Stokes' drift, the displacement of the particles from their original positions grows with time (e.g. Mei, 1989). Because of this feature, the method apparently cannot be applied to solve many important water wave problems.

The aim of this paper is to develop an asymptotic technique based on the Lagrangian description of fluid motion, which can be used to model nonlinear surface gravity waves for a wide range of problems of practical and theoretical interest, including travelling waves, diffraction problems, motion in closed basins, and so on.

2 LAGRANGIAN FORMULATION OF WATER WAVE PROBLEM

Fluid motion in the Lagrangian method is described by tracing the marked fluid particles. For two-dimensional motion we have $x = x(a, c, t); z = z(a, c, t)$, where (x, z) are Cartesian coordinates of the particle marked by Lagrangian indexes (a, c) at the time t. Due to the volume conservation of the incompressible fluid, the Jacobian J of the mapping $(x, z) \to (a, c)$ is motion invariant. The Lagrangian labels can be chosen in such a way that

$$J = \frac{\partial(x, z)}{\partial(a, c)} = 1. \tag{1}$$

The fluid occupies a single-connected domain in (x, z)-space, which is represented by a rectangle $a_1 \le a \le a_2; -h \le c \le 0$ in the space of Lagrangian indexes, where $c = -h$ corresponds to the rigid bed and $c = 0$ to the free surface. The side boundaries correspond to the rigid walls when a_1 and a_2 are finite, or to the free boundaries at $\pm\infty$.

The equations of motion of inviscid incompressible fluid in Lagrangian coordinates (a, c) can be obtained from Hamilton's variational principle (e.g. Herivel, 1955). Let us represent the Lagrangian density in the following form

$$\mathcal{L} = \mathcal{T} - \mathcal{U} + \rho\, P(a, c, t)\,(J - 1),$$

where the kinematic condition (1) is enforced by means of the Lagrange multiplier P, and ρ is the fluid density. The densities of the kinematic and potential energies of the fluid motion are

$$\mathcal{T} = \rho\,(x_t^2 + z_t^2)/2; \quad \mathcal{U} = \rho\, g\,(1 + A_z(t))\, z + \rho\,\alpha\,\omega_c^2 A_x(t)\, x.$$

The functions $A_x(t)$ and $A_z(t)$ take into account the Cartesian components of accelerations of the (x, z)-frame if we consider the fluid in a moving tank. The vertical acceleration $A_z(t)$ is scaled by the gravity acceleration g, and the horizontal one $A_x(t)$ by the characteristic acceleration of the fluid particles $\alpha\omega_c^2$, where α is the characteristic scale of particle displacement and ω_c is the characteristic frequency of the wave motion. According to Hamilton's principle the variation of the action integral must be zero. Taking the variation leads to the following equations describing the dynamics of the fluid particles

$$x_{tt} + \frac{\partial(P, z)}{\partial(a, c)} + \alpha\,\omega_c^2\, A_x(t) = 0; \quad z_{tt} + \frac{\partial(x, P)}{\partial(a, c)} + g\,(\,1 + A_x(t)\,) = 0, \tag{2}$$

plus the kinematic continuity condition (1). The Lagrange multiplier ρP can be recognised as pressure and the boundary condition on the free surface $c = 0$ is $P = 0$. Equations (2) can be resolved with respect to the spatial pressure derivatives and rewritten in the following form

$$\frac{\partial P}{\partial a} + g\,(1 + A_z(t))\, z_a + \alpha\,\omega_c^2 A_x(t)\, x_a = -x_{tt}x_a - z_{tt}z_a; \quad \frac{\partial P}{\partial c} + g\,(1 + A_z(t))\, z_c + \alpha\,\omega_c^2 A_x(t)\, x_c = -x_{tt}x_c - z_{tt}z_c. \tag{3}$$

The terms in the left hand sides of (3) are the components of gradient of a certain scalar function in the label space. Taking the curl of both sides of (3) we find that the value $\Omega = \nabla_a \times (x_t x_a + z_t z_a, x_t x_c + z_t z_c)$ is the invariant of the motion, that is $\partial\Omega/\partial t = 0$, where $\nabla_a\times$ is the curl operator in (a, c)-space. This is the Lagrangian form of vorticity conservation. If the fluid in the domain at initial time is irrotational we have the condition $\Omega = 0$, which can be written as

$$-\,\Omega = \frac{\partial(x, x_t)}{\partial(a, c)} + \frac{\partial(z, z_t)}{\partial(a, c)} = 0. \tag{4}$$

When the kinematic conditions (1) and (4), and the boundary conditions on the bottom and side boundaries are satisfied, it is sufficient to fulfil the first of equations (3) on the free surface $c = 0$ to define the flow in the whole domain. For our case of constant surface pressure this leads to the following dynamical condition:

$$x_{tt}x_a + z_{tt}z_a + g\,(1 + A_z(t))\, z_a + \alpha\,\omega_c^2 A_x(t)\, x_a\Big|_{c=0} = 0. \tag{5}$$

Thus, our aim is to construct functions (x, z) for continuous (1) irrotational (4) flow satisfying the dynamical free surface condition (5).

3 ASYMPTOTIC REPRESENTATION OF THE SOLUTION

Let us chose the Cartesian positions of the fluid particles at equilibrium as the Lagrangian indexes. Then the current particle position can be written as

$$x(a, c, t) = a + \alpha\,\xi(a, c, t); \quad z(a, c, t) = c + \alpha\,\zeta(a, c, t),$$

where the dimensionless functions ξ and ζ describe the particle displacement from equilibrium, and α is the characteristic scale of this displacement. We shall suppose α to be small compared to the scale of the wave motion with the characteristic wave number k: $k\alpha \to 0$. Substitution into (1) in the leading approximation gives the equation

$$\frac{\partial \xi}{\partial a} + \frac{\partial \zeta}{\partial c} = 0. \tag{6}$$

To satisfy the continuity at higher order we use the deformation of coordinates of the form $a_1 = a + \alpha\,\xi(a, c)/2$ and $c_1 = c + \alpha\,\zeta(a, c)/2$. Successively repeating this procedure, we finally obtain the following recursive asymptotic representation of the solution:

$$\begin{gathered} x(a, c, t) = a + \alpha\,\xi(a_n, c_n, t); \quad z(a, c, t) = c + \alpha\,\zeta(a_n, c_n, t) \\ a_0 = a; \quad c_0 = c; \quad a_n = a + \alpha\,\xi(a_{n-1}, c_{n-1}, t)/2; \quad c_n = c + \alpha\,\zeta(a_{n-1}, c_{n-1}, t)/2, \end{gathered} \tag{7}$$

where functions $\xi(a, c)$ and $\zeta(a, c)$ satisfy equation (6). Stopping the recursion at level n we satisfy continuity with accuracy $O(k\alpha)^{n+1}$. Equation (6) can be satisfied by using a single "stream function" Ψ:

$$\xi(a, c, t) = -\frac{1}{k}\frac{\partial \Psi}{\partial c}; \quad \zeta(a, c, t) = \frac{1}{k}\frac{\partial \Psi}{\partial a}. \tag{8}$$

To satisfy asymptotically the irrotationality condition (4) and the free surface condition (5) we expand the function Ψ into asymptotic series with powers of small parameter: $k\alpha\ \Psi = \Psi_0 + k\alpha\,\Psi_1 + \cdots$. Substituting this expansion into (4) we obtain in the leading approximation $\partial(\nabla^2\Psi_0)/\partial t = 0$. Thus, the function Ψ_0 is the sum of a harmonic function depending on time as a parameter and an arbitrary function of coordinates. This reflects the non-uniqueness of the choice of Lagrangian indexes (a, c). For our choice of the Lagrangian indexes the arbitrary function of coordinates is harmonic, and the whole function Ψ_0 satisfies Laplace equation $\nabla^2\Psi_0 = 0$. For higher approximations we obtain Poisson equations with the right-hand sides depending on the previous approximations. The details of the solution procedure depend on the particular problem in hand. Here we consider two examples of application of the approach described above.

4 APPLICATION FOR A STANDING WAVE

For the sloshing motion in a rectangular tank $0 < x < b,\ -h < z < \zeta(x,t)$ we can represent the solution for function Ψ_0 as an expansion by the linear sloshing modes (eigen modes). For our choice of Lagrangian indexes we can write

$$\Psi_0 = \sum_{n=1}^{\infty} \frac{1}{n}\frac{\sinh(nk(c+h))}{\sinh(nk\,h)}\sin(nk\,a)F_n(t),$$

which satisfies the Laplace equation in the rectangular domain $0 < a < b,\ 0 < c < -h$, and the boundary conditions on the rigid walls $a = 0;\ a = b$ and the bottom $c = -h$. Here k is the wavenumber of the first sloshing mode $k = \pi/b$, and functions F_n describe the time evolution of the individual modes. The right hand sides in the Poisson equations of higher approximations will include the products of derivatives of Ψ_0. To proceed we have to expand the right hand sides into series with respect to linear sloshing modes. The terms of these expansions include double infinite sums for the second order, triple sums for the third order, etc, which makes the analysis for the high orders extremely difficult. Nevertheless, in the case when there is one dominating mode (e.g. in the case of resonance) the procedure for constructing the higher order solutions becomes much simpler. Such a dominating mode will generate only a restricted number of modes at higher orders. We have found that the 3rd order solution for continuous irrotational flow can be represented in the following form

$$\begin{aligned}\Psi = \sum_{n=1}^{3}(k\alpha)^{n-1}\frac{\sinh(nk(c+h))}{n\,\sinh(nk\,h)}\sin(nk\,a)F_n(t) + (k\alpha)^2\frac{\sin(3ka)\sinh(k(c+h)) + \sin(ka)\sinh(3k(c+h))}{4\,\sinh(k\,h)\sinh(2k\,h)}F_{12}(t)+\\ + (k\alpha)^2\frac{\sin(3ka)\sinh(k(c+h)) + \sin(ka)\sinh(3k(c+h))}{32\,\sinh(k\,h)^3}F_{111}(t) + O(k\alpha)^3\end{aligned} \tag{9}$$

where k is the wavenumber of a dominating mode. The two last terms here are used to satisfy the kinematic irrotationality condition (4). The behaviour of these terms is completely defined by the behaviour of the eigen modes F_n through the following differential relations:

$$F_{12}'' = F_1'F_2 - F_1F_2'; \quad F_{111}'' = -F_1'F_1^2. \tag{10}$$

Functions F_n, describing the time evolution of eigen modes, can be found from the free surface boundary condition (5). Substituting (9) back into (7) and (5), we expand the resulting expression into a Taylor series for small $k\alpha$ up to the third order and collect terms proportional to $\sin(nka)$ with the same n. The terms due to the horizontal forcing are proportional to $A_x(t)$ and include the coefficients $\cos(mka),\ m = 0, 1, 2...$, which should be expanded into a Fourier series with respect to $\sin nka,\ n = 1, 2...$ on the interval $a \in [0, b]$. As a result we obtain a system of three non-linear differential equations describing the time evolution of modal functions $F_n(t),\ n = 1, 2, 3$. The final formulation consists of the initial value problem for ODEs, including the three equations for F_n together with equations (10) and proper initial conditions.

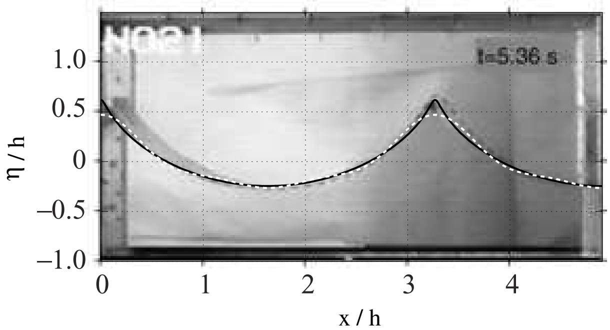
5 APPLICATION FOR A REGULAR TRAVELLING WAVE

To resolve the Stokes' drift problem, which arises in the case of open infinite domains and leads to solution being non-uniformly valid at large time, we represent the total motion of the particle as a steady motion with the mean drift velocity and small oscillations around the steadily moving mean position. Thus, we rewrite the recursive representation of the solution (7) in the following form:

$$\begin{aligned} x(a,c,t) = a + \alpha\,\omega t\,f(c) + \alpha\,\xi(a_n, c_n, t); \quad z(a,c,t) = c + \alpha\,\zeta(a_n, c_n, t)\\ a_0 = a + \alpha\,\omega t\,f(c); \quad c_0 = c; \quad a_n = a + \alpha\,\omega t\,f(c) + \alpha\,\xi(a_{n-1}, c_{n-1}, t)/2; \quad c_n = c + \alpha\,\zeta(a_{n-1}, c_{n-1}, t)/2,\end{aligned} \tag{11}$$

where functions ξ and ζ are to be expressed by using Ψ as in equations (8). The function Ψ describes the periodic part of the solution, and function $f(c)$ represents the variation of mean Stokes' drift with depth. We look for 5th order asymptotic expansions for unknown functions and frequency:

$$\Psi(a,c,t) = \textstyle\sum_{n=0}^{4}(k\alpha)^n\Psi_n(a,c,t) + O(k\alpha)^5;\ f(c) = \sum_{n=0}^{4}(k\alpha)^n f_n(c) + O(k\alpha)^5;\ \omega = \sum_{n=0}^{4}(k\alpha)^n\omega_n + O(k\alpha)^5.$$



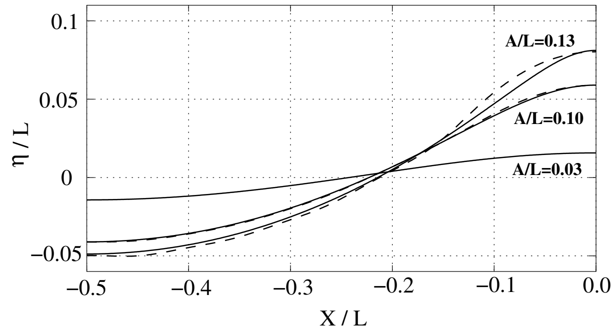


Figure 1: The comparison of the Lagrangian asymptotic solutions (solid lines) with the analogous Eulerian solutions (broken lines). Right: surface elevation profiles for a regular travelling wave. Left: surface elevation profiles for standing wave coupled with an experiment of Bredmose *et al.* (2003) (reproduced by kind permission of the authors).

The coefficients of the expansions can be found after back substitution into (11), (5) and (4), and expanding the result into Taylor series with respect to small $k\alpha$. Functions Ψ_n are $(c;t)$-periodic functions satisfying Laplace and Poisson equations, and are the linear combinations of terms $e^{m_1\,kc}\sin(m_2(ka+\omega t))$. For the case of deep water they are found to be

$$\Psi_0 = e^{kc}\sin(ka-\omega t);\ \Psi_1 = 0;\ \Psi_2 = \tfrac{5}{8}\,e^{3\,kc}\sin(ka-\omega t);\ \Psi_3 = (\tfrac{1}{4}\,e^{2\,kc} - \tfrac{5}{24}\,e^{4\,kc})\,\sin(2(ka-\omega t));$$
$$\Psi_4 = (\tfrac{3}{4}\,e^{3\,kc} + \tfrac{39}{32}\,e^{5\,kc})\,\sin(ka-\omega t) + (\tfrac{1}{36}\,e^{3\,kc} + \tfrac{49}{1152}\,e^{5\,kc})\,\sin(3(ka-\omega t)).$$

Functions f_n are used to balance the non-periodic terms of the equations, and the terms ω_n in the frequency expansion deal with the dynamic free-surface condition (5). We obtain

$$f(c) = (k\alpha)\,e^{2\,kc} + 2\,(k\alpha)^3\,e^{4\,kc} + O(k\alpha)^5;\ \omega = \sqrt{gk}\,(1 + \tfrac{1}{2}(k\alpha)^2 + \tfrac{9}{8}(k\alpha)^4 + O(k\alpha)^6).$$

The wave amplitude $A = (\,z(0,0,0) - z(\pi/k,0,0)\,)/2$ can be represented as an expansion with respect to small parameter $A = \alpha\,(1 + \frac{1}{2}(k\alpha)^2 + \frac{9}{8}(k\alpha)^4 + O(k\alpha)^6)$. We can introduce a new small parameter kA, which is more physically relevant and independent of the particular form of the solution representation. The expansion of the square of phase velocity with respect to this parameter is $C^2 = (\omega/k)^2 = gk\,(1 + (kA)^2 + \frac{1}{2}(kA)^4 + O(kA)^6)$, which coincides with the corresponding result obtained from Stokes' expansion.

6 RESULTS

The third order solution for sloshing in a 2D rectangular tank with a dominant mode and the fifth order asymptotic expansion for a plane regular wave in deep water were obtained. Some of the results are shown on figure 1. On the left there is a comparison of the 3rd order Lagrangian profile of a parametrically forced standing wave (black solid line) and the corresponding 3rd order Eulerian profile (white broken line) both plotted on top of the experiment of Bredmose *et al.* (2003), reproduced by kind permission of the authors. On the right the free surface profiles for regular travelling waves of various steepness obtained by 5th order new Lagrangian theory (solid lines) are compared with the 8th order classical Stokes' expansions (broken lines). It can be clearly seen that our Lagrangian predictions of the free surface are considerably better then those of the corresponding asymptotic solutions of the same orders in the Eulerian description. The great improvement of our new Lagrangian perturbation method compared to early attempts is because our solution for a regular wave is uniformly valid for all times. In general, the results obtained demonstrate that the new theory provides a good description of nonlinear water waves and has certain advantages compared to the classical Eulerian perturbation theory. The authors believe that further development of the approach will lead to a simple method relevant for many practical engineering applications.

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Discusser: H. Bredmose

This sounds very interesting. To apply the method to a specific problem, is it necessary to run a symbolic processor for each new case? What is the computational time for the Bredmose et al. (2003) experiments?

Author's reply:

A symbolic processor helps to obtain an asymptotic formulation for a specific class of problems. After this the simplified formulation can be applied for solving various problems within this class, for example, for various tank sizes and motions in the presented problem on the forced sloshing in a rectangular tank. The numerical solution of the problem in final asymptotic formulation is very fast. For the cases in the paper computations take only few seconds and can be performed in real time.

Discusser: Q.W. Ma

From your presentation we saw the applications of the new method to water wave problems without floating bodies. I would like to know if the method could be applied to the problems with floating bodies on the free surface.

Author's reply:

The basic concept of the method is quite general and the only principal restriction is that the flow should be two dimensional. Nevertheless, for every new class of problems an extensive work has to be done to obtain final asymptotic formulation. The authors believe that it is possible and for certain problems including floating bodies as well.