Response of unsteady external load on the elastic circular plate floating on shallow water

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1. ABSTRACT

A behavior of a floating elastic circular plate under unsteady external loading is studied. It is assumed that the depth of the fluid is small in comparison with the radius of the plate, and the shallow water approximation is used. The edges of the plate are free. A combined motion of the elastic plate with the fluid is considered within the linear theory. The plate deflections for some unsteady external loads have been calculated. The hydroelastic responses of the infinite plate and the bounded one to a unsteady load differ essentially.

1. INTRODUCTION

The action of dynamic loads on thin floating plates is being studied intensively as applied to the infinite ice sheets (see *e.g.* Squire *et.al.* (1996)). In addition, interest in the unsteady behavior of floating elastic structures has increased recently because of the design of floating platforms for various purposes. The operation of these platforms requires determination of their dynamic properties with respect to the action of unsteady external loading due to intense traffic, load movement, takeoffs and landings of airplanes, missile takeoffs, *etc.* The solution of unsteady 3-D hydroelastic problems is a difficult task even for linear formulation and requires high computational cost (see *e.g.* Kashiwagi (2000)).

In this paper, a simplified model is proposed in which the depth of the fluid is assumed to be smaller than the horizontal dimensions of the plate, and the shallow water approximation is used.

In the 2-D case unsteady response of an elastic beam floating on a shallow water under external load with the vertical inertia of the moving mass was considered by Sturova (2002b). It is shown that the vertical inertia of the moving mass must be taken into account only for the massive load and the large acceleration.

1. MATHEMATICAL FORMULATION

The circular elastic plate of radius r_0 is freely floating on a fluid-layer having a constant density ρ and depth H. The surface of the fluid that is not covered with the plate is free. The fluid is assumed to be incompressible and inviscid, and its flow is irrotational. The velocity potentials describing the fluid motion in the regions under the plate and outside the plate are denoted by $\phi^{(1)}(x,y,t)$ and $\phi^{(2)}(x,y,t)$, respectively. Here x and y are the horizontal coordinates with the origin in the center of the elastic plate, t is time.

The plate deflection $w(x,y,t)$ is defined by the equation

$$D\Delta^2 w + \rho_1 h_1 \frac{\partial^2 w}{\partial t^2} + g\rho w + \rho \frac{\partial \phi^{(1)}}{\partial t} = -P(x,y,t) \quad (r \le r_0), \tag{1}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2,\ r = \sqrt{x^2+y^2},\ D = Eh_1^3/[12(1-\nu^2)],\ E,\ \rho_1,\ h_1,\ \nu$ are the effective Young's modulus, the density, the thickness, and Poisson's ratio of the plate, g is the acceleration due to gravity. The function $P(x,y,t)$ is a prescribed function, and it describes the external pressure acting upon the plate that is independent of the plate motion (so-called inertia-free loading).

According to linear shallow-water theory, the following relation is valid:

$$\frac{\partial w}{\partial t} = -h\Delta\phi^{(1)} \quad (r \le r_0), \quad h = H - d. \tag{2}$$

Here $d = \rho_1 h_1/\rho$ is the draft of the plate.

In the free-water region, the velocity potential $\phi^{(2)}(x,y,t)$ satisfies the equation

$$\frac{\partial^2 \phi^{(2)}}{\partial t^2} = gH\Delta\phi^{(2)} \quad (r > r_0), \tag{3}$$

with the condition of damping out of the motion $\nabla\phi^{(2)} \to 0$ at $r \to \infty$.

It is also of interest to solve this problem under the assumption of a non-gravity fluid at $r > r_0$. This model is used in the theory of impact to study short-duration external action on a floating elastic body (Korobkin (2000))

$$\phi^{(2)}(x,y,t) = 0 \quad (r > r_0). \tag{4}$$

If $r = r_0$, the following matching conditions (continuity of pressure and mass flow) must be satisfied:

$$\frac{\partial \phi^{(1)}}{\partial t} = \frac{\partial \phi^{(2)}}{\partial t}, \qquad \frac{\partial \phi^{(1)}}{\partial r} = \frac{H}{h}\frac{\partial \phi^{(2)}}{\partial r}. \tag{5}$$

At the edges of the plate, the free-edge conditions are satisfied, which imply that the bending moment and shear force are equal to zero:

$$\Delta w - \frac{\nu_1}{r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial w}{\partial r}\right) = \frac{\partial \Delta w}{\partial r} + \frac{\nu_1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right) = 0 \quad (r = r_0),$$

where $\theta = \mathrm{arctg}(y/x), \quad \nu_1 = 1 - \nu$.

We assume that at the initial time, the fluid and the plate are at rest:

$$w = \frac{\partial w}{\partial t} = \phi^{(1)} = \phi^{(2)} = \frac{\partial \phi^{(2)}}{\partial t} = 0 \quad (t = 0). \tag{6}$$

Non-dimensional variables are used below: r_0 is taken as the length scale and $\sqrt{r_0/g}$ as the time scale. Then, we use the following non-dimensional coefficients

$$\beta = \frac{H}{h}, \quad \gamma = \frac{d}{r_0}, \quad \delta = \frac{D}{\rho g r_0^4}.$$

3. EIGENFUNCTION EXPANSION

Let us assume for simplicity that the external pressure in eqn (1) is the even function on y and the plate deflections are symmetric about the x-axis. In the non-dimensional variables, we seek the plate deflection as the sum of eigenfunctions of a circular thin plate of unit radius with the free edge conditions

$$w(r,\theta,t) = \sum_{n=0}^{\infty} \cos n\theta \sum_{j=0}^{\infty} a_{jn}(t) W_{jn}(r). \tag{7}$$

Here the functions $a_{jn}(t)$ are to be determined, and the functions $W_{jn}(r)$ are the solutions of the eigenvalue problem

$$L_n W_{jn} = \lambda_{jn}^4 W_{jn} \quad (r \le 1), \quad L_n \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2},$$

$$W_{jn}'' + \nu W_{jn}' - n^2\nu W_{jn} = (L_n W_{jn})' + n^2\nu_1(W_{jn} - W_{jn}') = 0 \quad (r = 1),$$

where the prime denotes derivation with respect to r. This eigenvalue problem is well studied (see *e.g.* Itao & Crandall (1979)). There are two rigid body modes $W_{00} = \sqrt{2}$ and $W_{01} = 2r$. The other eigenfunctions have the form

$$W_{jn} = A_{jn}[J_n(\lambda_{jn} r) + C_{jn} I_n(\lambda_{jn} r)].$$

The J_n are Bessel functions of the first kind and the I_n are modified Bessel functions of the first kind. The frequency parameter λ_{jn} and the mode shape parameter C_{jn} are fixed by the eigenvalue problem, and the amplitude parameter A_{jn} is fixed by the normalization requirement

$$\int_0^1 r W_{jn}(r) W_{kn}(r) \mathrm{d}r = \delta_{jk},$$

where δ_{jk} is the Kronecker delta.

Since we are considering the symmetric problem about the x-axis, the potentials $\phi^{(1,2)}(r,\theta,t)$ may be written as

$$\phi^{(1,2)}(r,\theta,t)=\sum_{n=0}^{\infty}\Phi_n^{(1,2)}(r,t)\cos n\theta. \tag{8}$$

Using eqns (7), (8), we seek the solution for $\Phi_n^{(1)}(r,t)$ in the form which satisfies eqn (2)

$$\Phi_n^{(1)}(r,t)=-\frac{1}{h}\left\{\sum_{j=0}^{\infty}\dot{a}_{jn}(t)[\Psi_{jn}(r)-r^n\Psi_{jn}(1)]+r^n q_n(t)\right\}, \tag{9}$$

where an overdot denotes derivation with the respect to time, and the functions $\Psi_{jn}(r)$ are determined from the equation

$$L_n\Psi_{jn}(r)=W_{jn}(r).$$

For the rigid body modes they are equal to

$$\Psi_{00}(r)=\frac{r^2}{2\sqrt{2}},\quad \Psi_{01}(r)=\frac{r^3}{4},$$

and for other modes we have

$$\Psi_{jn}=A_{jn}[C_{jn}I_n(\lambda_{jn}r)-J_n(\lambda_{jn}r)]/\lambda_{jn}^2.$$

The functions $q_n(t)$ in eqn (9) are to be determined. Assuming that the fluid is non-gravity for $r>1$ and using eqn (4), we have $q_n\equiv 0$. With regard to gravity these functions are determined from the matching conditions for the potentials $\Phi_n^{(1,2)}(r,t)$ and their derivatives with respect to r following from eqn (5)

$$\Phi_n^{(1)}=\Phi_n^{(2)},\quad \frac{\partial\Phi_n^{(1)}}{\partial r}=\beta\frac{\partial\Phi_n^{(2)}}{\partial r}\quad (r=1).$$

According to eqn (3), the equation for $\Phi_n^{(2)}(r,t)$ has the form

$$\frac{\partial^2\Phi_n^{(2)}}{\partial t^2}=HL_n\Phi_n^{(2)}\quad (r>1),\quad \Phi_n^{(2)}\to 0\quad (r\to\infty).$$

Using the notations

$$\Phi_n^{(2)}|_{r=1}=S_n(t),\quad \frac{\partial\Phi_n^{(2)}}{\partial r}|_{r=1}=T_n(t),$$

and applying Laplace transform, we obtain the relation

$$S_n(t)=\sqrt{H}\int_0^t T_n(\tau)G_n(\sqrt{H}(t-\tau))\mathrm{d}\tau. \tag{10}$$

Here so-called transition function $G_n(\xi)$ has the form

$$G_n(\xi)=\frac{1}{2i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{K_n(s)}{sK_n'(s)}\exp(s\xi)\mathrm{d}s.$$

The K_n are modified Bessel functions of the second kind. The functions $G_n(\xi)$ are tabulated up to $n=10$ by Randall (1958), and their simple approximations are given by Leclerc (1970).

We substitute eqns (7)-(9) into non-dimensional analog of eqn (1) and the initial conditions (6), multiply the obtained relations by $r\cos(n\theta)W_{kn}(r)$, and integrate them over r from 0 to 1 and over θ from 0 to 2π. Using the properties of the functions $W_{kn}(r)$, we obtain the system of ordinary differential equations (ODE)

$$\sum_{j=0}^{\infty}\ddot{a}_{jn}\left[\gamma\delta_{kj}-\frac{1}{h}D_{kj}^{(n)}\right]+(1+\delta\lambda_{kj}^4)a_{kn}-\frac{1}{h}S_{kn}\dot{q}_n=-P_{kn}\quad (k=0,1,2...) \tag{11}$$

with the initial conditions

$$a_{kn}(0) = \dot{a}_{kn}(0) = q_n(0) = 0.$$

Here

$$D_{kj}^{(n)} = \int_0^1 rW_{kn}(r)[\Psi_{jn}(r) - r^n\Psi_{jn}(1)]\mathrm{d}r, \quad D_{kj}^{(n)} = D_{jk}^{(n)}, \quad S_{kn} = \int_0^1 r^{n+1}W_{kn}(r)\mathrm{d}r,$$

$$P_{kn}(t) = \frac{\varepsilon_n}{\pi}\int_0^\pi \cos(n\theta)\mathrm{d}\theta \int_0^1 rP(r,\theta,t)W_{kn}(r)\mathrm{d}r, \quad \varepsilon_0 = 1, \varepsilon_n = 2 \ (n \geq 1).$$

The system of ODE is closed by the integro-differential equation, which is a consequence of eqn (10)

$$q_n(t) = \frac{h}{\sqrt{H}}\int_0^t \left[\sum_{j=0}^{\infty} \dot{a}_{jn}(\tau)b_{jn} + nq_n(\tau)\right] G_n(\sqrt{H}(t-\tau))\mathrm{d}\tau, \tag{12}$$

where

$$b_{jn} = \frac{\partial \Psi_{jn}}{\partial r}|_{r=1} - n\Psi_{jn}(1).$$

It is easy to show, that $S_{kn} = b_{kn}$. The system of integro-differential equations (11), (12) is simplified at $n = 0, 1$, where all values S_{kn} are equal to zero, except $S_{00} = 1/\sqrt{2}$ and $S_{01} = 1/2$.

Using the reduction method, the infinite sets in eqns (7), (8), (12) are replaced by finite sums. The linear system of eqns (11), (12) is solved by method of finite differences with uniform time steps. For the calculation of the convolution integral in (12) we used the method proposed by Kashiwagi (2000).

To test the computational algorithm, we used the solution of the steady-state problem of the action of time-harmonic surface pressures on an elastic circular plate floating on shallow water (Sturova (2002a)). The calculations for a circular plate were compared with the well-known solution for an infinite plate.

4. DISCUSSIONS

The plate deflections for some unsteady external loads have been calculated and will be presented at the Workshop. The hydroelastic responses of the infinite plate and the bounded one to a unsteady load differ essentially. As was noted previously for time-harmonic external loading (Sturova (2002a)) and for unsteady loading of the elastic beam (Sturova (2002b)), in some cases, the amplitudes of oscillations of the plate at its edges exceed the corresponding values for the middle region of the plate.

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