FARFIELD WAVES

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1. Introduction

An offshore structure in regular (time-harmonic) ambient waves generates a single system of circular waves centered at the offshore structure. A ship advancing with constant speed along a straight path in calm water does not create waves outside a wedge with included angle of 39^o (in deep water) aft the ship bow, and produces two systems of waves—called transverse and divergent waves—inside the 39^o wedge. A ship advancing in regular (time-harmonic) ambient waves generates much more complex systems of waves. The characteristics of these wave systems are determined by the parameter $\tau = \mathcal{U}\,\omega/g$, where $\mathcal{U}$ is the ship speed, ω is the frequency of the waves encountered by the ship, and g is the gravitational acceleration. If $\tau < 1/4$, the free surface can be divided into three regions: there is a region where a single system of ring waves, approximately elliptical in shape, exists; three systems of waves—the already-mentioned ring waves, and a system of transverse and divergent waves—are created in another region in the shape of a wedge trailing the ship bow; the third region is a smaller wedge where five systems of waves—the three wave systems that exist in the second region, and another system of transverse and divergent waves—are found. If $\tau > 1/4$, we have a region of the free surface where no waves are created, another region where two systems of waves exist, and a third region where four systems of waves are observed. Thus, widely different systems of waves, with characteristics (wavelength, celerity, direction of propagation) that vary widely, are found in free-surface hydrodynamics of ships and offshore structures. These wave systems have been extensively studied in the literature; see e.g. *Wehausen and Laitone (1960), Whitham (1974), Lighthill (1978).*

This study shows that these diverse and relatively complicated wave systems can be defined by a single, remarkably simple, generic mathematical representation that can be obtained very simply, using elementary analysis based only on basic concepts. This representation of farfield waves provides considerable information (including wave patterns, wavelength, wave celerity) and shows that—except for their amplitudes—farfield waves are entirely determined by the dispersion relation associated with the class of dispersive waves under consideration. Indeed, although farfield waves generated by a ship advancing in regular waves are primarily considered here, the approach and the analysis are valid more generally and in fact can easily be applied to a broad class of dispersive waves.

The study also presents a farfield representation that defines time-harmonic (or steady) farfield ship waves in terms of one-dimensional Fourier superpositions of elementary waves that satisfy the Laplace equation, the boundary conditions at the sea floor and the free surface, and the radiation condition. This Fourier representation of farfield waves can be regarded as a generalization of two representations given in the literature: a classical representation that satisfies the Laplace equation and the sea-floor and free-surface boundary conditions but does not satisfy the radiation condition, and the farfield representation given in *Noblesse and Chen (1995)*, which satisfies the radiation condition and the sea-floor and free-surface boundary conditions but does not satisfy the Laplace equation in the nearfield.

2. Problem statement

A ship advances in regular (time-harmonic) waves along a straight path, with constant speed $\mathcal{U}$, at the free surface of a large body of water of uniform depth D . The X axis is chosen along the path of the ship, and points toward the ship bow. The Z axis is vertical and points upward, and the mean free surface is taken as the plane $Z=0$. The sea floor $Z=-D$ is assumed to be a rigid wall. The flow is observed from a Cartesian system of coordinates moving with speed $\mathcal{U}$ along the X axis, and is expressed as the sum of a steady component, which represents the flow due to the ship advancing in calm water, and a time-harmonic component associated with the ambient waves, the waves diffracted by the ship, and the radiated waves due to the motions of the ship about a mean position. g and ω represent the gravitational acceleration and the frequency of the waves encountered by the ship, respectively. The steady and time-harmonic components of the flow due to the ship are expressed in nondimensional form in terms of the water density and a reference length L (typically the ship length) and velocity U, which can be taken as $\sqrt{gL}$ or as the ship speed $\mathcal{U}$. Thus, nondimensional coordinates $\vec{\xi} = (\,\xi\,,\eta\,,\zeta\,) = \vec{X}/L$ and velocity potential $\phi = \Phi/(UL)$ are used here. The nondimensional water depth is $d = D/L$.

Farfield waves generated by a ship may be analyzed within the framework of linear potential flow, and thus are defined in terms of a velocity potential ϕ that satisfies the Laplace equation, the boundary condition $\phi_\zeta = 0$ at the (rigid) sea-floor $\zeta = -d$, and the free-surface boundary condition

$$\phi_\zeta + F^2 \phi_{\xi\xi} - f^2 \phi + i\, 2\tau\, \phi_\xi - \varepsilon (F \phi_\xi + i f \phi) = 0 \quad \text{at } \zeta = 0 \tag{1}$$

where $F = \mathcal{U}/\sqrt{gL}$, $f = \omega\sqrt{L/g}$, $\tau = Ff$ and $0 < \varepsilon \ll 1$. The potential of farfield steady ship waves satisfies a free-surface condition that corresponds to the special case $f = 0 = \tau$ of the condition (1) associated with diffraction-radiation of time-harmonic waves with forward speed. The parameter ε in the free-surface boundary condition (1) is associated with initial conditions that correspond to a flow starting from rest at time $T = -\infty$; see e.g. *Lighthill (1978)* or *Noblesse (2001).* Finally, the potential ϕ satisfies a boundary condition at the ship hull that specifies the derivative $\partial\phi/\partial n$ of ϕ along the normal to the hull surface. However, this nearfield boundary condition is not required for the present analysis.

3. Case $\varepsilon = 0$: classical representation of farfield waves

The elementary wave function

$$W(\xi, \eta, \zeta) = e^{-i(\alpha\xi + \beta\eta)} \cosh[\, k\, (\zeta + d\,)\,] \,/\, \cosh(kd\,) \tag{2a}$$

satisfies the sea-floor condition, and satisfies the Laplace equation if

$$k = \sqrt{\alpha^2 + \beta^2} \tag{2b}$$

The elementary wave function (2a) also satisfies the free-surface boundary condition (1), where ε is taken equal to 0 for now (the case $0 < \varepsilon \ll 1$ is examined further on), if the three parameters α, β, k satisfy the condition $D = 0$, known as the dispersion relation, with the dispersion function D given by

$$D = k \tanh(kd) - (F\alpha - f)^2 \tag{3}$$

If (2b), which defines the wavenumber k in terms of α and β is used in (3), the dispersion function $D(\alpha, \beta, k)$ becomes a function $D(\alpha, \beta)$ of the two Fourier variables α and β, and the dispersion relation $D = 0$ defines one or several curves, called dispersion curves, in the Fourier plane (α, β).

Thus, the elementary wave function (2a), where α and β satisfy the dispersion relation $D = 0$ with $D(\alpha, \beta)$ defined by (3) and (2b), is an elementary farfield solution that satisfies the Laplace equation and the boundary conditions at the sea floor and the free surface. Accordingly, farfield waves can be represented by one-dimensional Fourier superpositions of elementary waves W:

$$\sum_{D=0} \int_{D=0} ds\; A\, W \tag{4}$$

Here, summation is performed over all the dispersion curves defined by the dispersion relation $D = 0$, the Fourier variables (α, β) lie on a dispersion curve, ds represents the differential element of arc length of a dispersion curve, W is the elementary wave function (2a), and A stands for an amplitude function (that is determined by the nearfield boundary condition at the ship hull surface). The farfield waves defined by the classical Fourier representation (4), associated with the free-surface boundary condition (1) with $\varepsilon = 0$ and the related dispersion relation $D = 0$, do not satisfy the radiation condition. Indeed, the representation (4) does not yield correct wave patterns; for instance, this classical representation of farfield waves does not preclude steady ship waves ahead of a ship advancing in calm water.

4. Case $0 < \varepsilon \ll 1$: farfield waves that satisfy the radiation condition

Elementary waves that correspond to the free-surface condition (1) with $0 < \varepsilon \ll 1$ can be defined by considering complex Fourier variables $\alpha + i\varepsilon\alpha_1, \beta + i\varepsilon\beta_1, k + i\varepsilon k_1$ in (2a), which thus becomes

$$e^{-i[(\alpha + i\varepsilon\alpha_1)\xi + (\beta + i\varepsilon\beta_1)\eta]} \cosh[\,(k + i\varepsilon k_1)(\zeta + d\,)\,] \,/\, \cosh[\,(k + i\varepsilon k_1)\, d\,] \tag{5a}$$

This elementary wave satisfies the sea-floor condition, and satisfies the Laplace equation if the conditions

$$k = \sqrt{\alpha^2 + \beta^2} \qquad k\, k_1 = \alpha\,\alpha_1 + \beta\,\beta_1 \qquad k_1 = \sqrt{\alpha_1^2 + \beta_1^2} \tag{5b}$$

are satisfied. These three conditions require

$$(\alpha_1, \beta_1, k_1) = \Gamma\, (\alpha, \beta, k) \tag{5c}$$

where Γ stands for a proportionality factor between α_1, β_1, k_1 and α, β, k. The elementary wave (5a) also satisfies the free-surface condition (1), with $0<\varepsilon\ll 1$, if $D=0$ and $k_1 D_k + \alpha_1 D_\alpha + D_1 = 0$. Here,

$$D_1 = F\alpha - f \tag{6a}$$

and $D_k = \tanh(kd) + kd/\cosh^2(kd)$ and $D_\alpha = -2\,(F^2\alpha - \tau)$ are the derivatives of the dispersion function $D(\alpha\,,\beta\,,k)$ with respect to k and α. If (5b) are used in the foregoing dispersion relations, we obtain

$$D = 0 \qquad\qquad \alpha_1 D_\alpha + \beta_1 D_\beta + D_1 = 0 \tag{6b}$$

where D_α and D_β now stand for the derivatives of the dispersion function $D(\alpha\,,\beta)$ defined by (3) with (2b). Expressions (5c) and (6b) define Γ as

$$\Gamma = \frac{-D_1}{\alpha D_\alpha + \beta D_\beta} = \frac{-D_1}{k D_k} = \frac{-\mu}{\Delta} \qquad \text{with} \qquad \Delta = k\,\frac{|D_k|}{|D_1|} \qquad \text{and} \qquad \mu = \text{sign}D_1\,\text{sign}D_k \tag{7}$$

Here, $D_k = D_\alpha\,\alpha/k + D_\beta\,\beta/k$ is the derivative of the dispersion function $D(\alpha\,,\beta)$ in the radial direction $(\alpha, \beta)/k$. The dispersion relations (3) and (6) correspond to the limit $\varepsilon \to 0$ of the dispersion relation

$$D(\alpha + i\,\varepsilon\,\alpha_1\,, \beta + i\,\varepsilon\,\beta_1) + i\,\varepsilon\,D_1(\alpha + i\,\varepsilon\,\alpha_1\,, \beta + i\,\varepsilon\,\beta_1) = 0 \tag{8}$$

associated with the free-surface boundary condition (1).

The elementary wave function defined by (5a), (5c) and (7) as

$$e^{-\,i\,(\,\alpha\,\xi\,+\,\beta\,\eta\,)\,-\,\mu\,\varepsilon\,(\,\alpha\,\xi\,+\,\beta\,\eta\,)/\Delta}\,\cosh\left[\,k\,(1 - i\,\mu\,\varepsilon/\Delta)(\zeta + d\,)\,\right]/\cosh\left[\,k\,(1 - i\,\mu\,\varepsilon/\Delta)\,d\,\right]$$

is unbounded in the farfield limit $\xi^2 + \eta^2 \to \infty$ if $\mu\,\text{sign}(\alpha\,\xi + \beta\,\eta) < 0$. The polar representations

$$(\,\alpha\,,\beta\,) = k\,(\,\cos\gamma, \sin\gamma\,) \qquad\qquad (\,\xi\,,\eta\,) = h\,(\,\cos\theta, \sin\theta\,)$$

of the Fourier variables α, β and the horizontal coordinates ξ, η and expression (7) for μ then show that bounded elementary waves are obtained if the condition $\text{sign}D_1\,\text{sign}D_k = \text{sign}(\alpha\,\xi + \beta\,\eta) = \text{sign}\cos(\gamma - \theta)$ is satisfied. This condition yields

$$\theta - \text{sign}D_1\,\text{sign}D_k\;\pi/2 \le \gamma \le \theta - \text{sign}D_1\,\text{sign}D_k\;\pi/2 + \pi \tag{9a}$$

The condition (9a) defines "acceptable" sections (that depend on θ) of the dispersion curves $D = 0$ in the classical representation (4) of farfield waves. Alternatively, the representation (4) can be modified as

$$\sum_{D=0} \int_{D=0} ds\; A\,\Lambda\,W \qquad \text{with} \qquad \Lambda = \left\{ \begin{matrix} \text{sign}D_1\,\text{sign}D_k + \text{sign}(\alpha\,\xi + \beta\,\eta) \\ \text{sign}D_1 + \text{sign}D_k\;\text{sign}(\alpha\,\xi + \beta\,\eta) \end{matrix} \right\} \tag{9b}$$

The farfield representation (4), where the dispersion curves $D = 0$ are restricted as specified by (9a), and the representation (9b) restrict the dispersion curves in equivalent ways. The representation (9b), with

$$\Lambda = \text{sign}D_1 + \text{sign}(D_\alpha\,\xi + D_\beta\,\eta) \tag{9c}$$

was obtained in *Noblesse and Chen (1995)* using a different, considerably more complicated, approach based on a farfield asymptotic analysis of a singular double Fourier integral that accounts for nearfield free-surface effects. Straightforward stationary-phase considerations, given in *Noblesse and Yang (2002)*, show that the Fourier representations of farfield waves associated with expressions (9c) and (9b) for the function Λ yield asymptotically equivalent farfield waves, as one expects.

The main contribution to the Fourier integral (9b) in the farfield limit $h \to \infty$ is known to stem from points where the phase $\alpha\,\xi + \beta\,\eta$ of the elementary wave function (2a) is stationary. It is easily shown that a point of stationary phase yields a non-zero contribution to (9b) if

$$(\,\xi\,,\eta\,)\,/\sqrt{\xi^2 + \eta^2} = (\,\cos\theta, \sin\theta\,) = \text{sign}D_1\,(\,D_\alpha\,,D_\beta\,)\,/\sqrt{D_\alpha^2 + D_\beta^2} = \text{sign}D_1\,\nabla D/\|\nabla D\| \tag{10}$$

The relation (10) shows that a point $(\alpha\,,\beta)$ of a dispersion curve (in the Fourier plane) mostly generates waves (in the physical space) in a direction $\eta/\xi = \tan\theta$ that is orthogonal to the dispersion curve and oriented as $(\mathrm{sign}D_1)\,\nabla D$. Conversely, farfield waves observed along a direction θ stem mostly from the point(s) of the dispersion curve(s) where the condition (10) holds. This well-known result, e.g. see *Lighthill (1978)* and *Chen and Noblesse (1997)*, provides a verification of the representation (9b), and also shows that this farfield approximation is asymptotically equivalent to the representation (9b), with Λ given by (9c), obtained in *Noblesse and Chen (1995)* using a different approach, as already noted.

5. Applications

The representation (9) is easily shown to provide considerable information about important wave characteristics (including wavelength, celerity, direction of propagation, and patterns), as shown in *Chen and Noblesse (1997)*. For instance, farfield wave patterns can be constructed from the dispersion functions D and D_1 in the dispersion relation (8) using the parametric equations

$$(\,\xi_n\,,\eta_n\,) = \mathrm{sign}D_1\,(\,D_\alpha\,,D_\beta\,)\,2n\,\pi/(\,k\,|D_k|\,) \tag{11}$$

These equations readily show that a point $(k\,,\gamma)$ of a dispersion curve where $D_k=0$ yields $\xi_n^2+\eta_n^2=\infty$, which corresponds to an asymptote (at an angle θ orthogonal to the angle γ) of the wave pattern. Other important wave-pattern features are easily determined from the dispersion curves. For instance, a cusp in a wave pattern corresponds to an inflection point of a dispersion curve.

The functions D and D_1 in the representation (9) of farfield waves stand for generic dispersion functions. This generic representation can easily be applied to the specific dispersion functions (3) and (6a) associated with the free-surface boundary condition (1) for time-harmonic flows with forward speed. The deep-water limit is briefly considered here. The dispersion curves defined by the dispersion relation (3) can be decomposed into three branches within which $\mathrm{sign}D_1$ and $\mathrm{sign}D_k$ are constant. These three branches and the corresponding values of $\mathrm{sign}D_1$ and $\mathrm{sign}D_k$ are given by

$$\left\{\begin{array}{lll} -\pi/2<\gamma<\pi/2\ : & \mathrm{sign}D_1=1 & \mathrm{sign}D_k=-1 \\ \pi/2<\gamma\le\pi-\gamma^\tau\ \cup\ \pi+\gamma^\tau\le\gamma<3\,\pi/2\ : & \mathrm{sign}D_1=-1 & \mathrm{sign}D_k=-1 \\ -\pi+\gamma^\tau\le\gamma\le\pi-\gamma^\tau\ : & \mathrm{sign}D_1=-1 & \mathrm{sign}D_k=1 \end{array}\right\} \tag{12}$$

with $0\le\gamma^\tau<\pi/2$ defined as $\gamma^\tau=0$ if $\tau\le 1/4$ and $\gamma^\tau=\cos^{-1}(1/4\tau)$ if $1/4\le\tau$. These three branches are defined by the complementary polar representations

$$F^2k=(\sqrt{1/4+\tau\cos\gamma}\,+1/2\,)^2/\cos^2\gamma \qquad k/f^2=1/(\sqrt{1/4+\tau\cos\gamma}\,+1/2\,)^2$$

where the second equation defines the third branch and the first equation defines the first two branches.

6. Nearfield-extension

The farfield representation (9) does not satisfy the Laplace equation in the nearfield. However, (9) can be modified as shown in *Noblesse and Yang (2002)*. Specifically, the function Λ in (9b) becomes

$$\Lambda=\mathrm{sign}D_1+\,\mathrm{sign}D_k\,\frac{\sinh(\Phi/\sigma)+i\,T\sin(V/\sigma)}{\cosh(\Phi/\sigma)+\cos(V/\sigma)} \tag{13}$$

where $\Phi=\alpha\,\xi+\beta\,\eta\,,V=k\,(\,\zeta+d\,)\,,T=\tanh V$ and σ is a positive real number (e.g. $\sigma=1$). In deep water, we have $V=k\,\zeta$ and $T=1$. Expression (13) is mathematically equivalent to the expression given in *Noblesse and Yang (2002)* but simpler and in the same form as (9c).

The farfield representation (9b) with Λ given by (13) satisfies the Laplace equation, the sea-floor and free-surface boundary conditions, and the radiation condition. This flow representation can be useful for numerical methods based on a spectral approach, to couple a nearfield calculation method and a farfield linear potential flow, and to define simple *farfield* Green functions that satisfy the free-surface boundary condition exactly in the farfield and approximately in the nearfield, as will be shown elsewhere.

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Question by : M. Tulin
What is the relation between your final representation (introducing the amplitude function A) and Havelock's original representation of the far field in the steady case (in which he invented the amplitude function)?

Author's reply:
As far as I am able to remember Havelock's paper (I will check my memory upon my return) his representation of far-field waves corresponds to the classical one given in the paper: i.e the radiation condition is not automatically satisfied in this classical representation; the radiation condition is satisfied in the far field wave representation given in the paper. The amplitude function A merely corresponds to a linear superposition of elementary waves and I do not know who invented the principle of linear superposition (I know it is not me).
