A Single-Integral Representation for the Green Function of Steady Ship Flow in Water of Finite Depth

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Compared to the deep water case, there are very few works devoted to the Green function of the steady ship flow in water of finite depth. One classical formulation of this Green function is given in Wehausen (1973). Its non-uniformity when the water depth tends to infinity is highlighted in Chen & Zhao (2001) who obtain a new uniform formulation by removing constant terms, one of which is infinite in magnitude. Both of these formulations contain a double integral which is difficult to evaluate numerically. In the deep water case, this problem is alleviated by the development of single-integral representations for the Green function. Some of these representations, such as the three given in Noblesse (1981), are much more amenable to numerical computation. It is desirable to derive similar single-integral expressions for the finite-depth case. Unfortunately, the approaches in Noblesse (1981) cannot be extended to finite depth. In this paper, we employ the technique used in Smorodin (1972) to reduce the double integral of the finite-depth Green function to single integrals by treating the θ -integration as the inner integration and evaluating it using Cauchy's residue theorem. The resulting single-integral representation contains infinite series which converge slowly for $kR >> 1$. In this case, three alternative expressions for the series are developed, and they are well suited to numerical computation.

1. Steady, finite-depth Green function

Using one of the forms of the ship-motion Green function developed in Chen & Nguyen (2000), we can readily obtain an expression for the steady Green function by taking the limit of the former as the frequency approaches zero. Introducing a coordinate system moving with the source at a velocity U along the positive x-axis where the (x, y)-plane coincides with the mean free surface and the positive z-axis points upward, we can write the steady Green function for a source at (x_s, y_s, z_s) and a field point at (ξ, η, ς) as the sum

$$G = G^S + G^F \tag{1}$$

where G^F accounts for the free-surface effects, and G^S is defined in terms of simple singularities as follows

$$4\pi G^S = \sum_{n=-\infty}^{\infty}(-1)^n\left[-1/\sqrt{r^2+(\zeta-z_s+2n)^2}+1/\sqrt{r^2+(\zeta+z_s+2n)^2}\right] \tag{2}$$

Here, we have normalized the variables by the water depth H and defined $r^2 = x^2 + y^2$ where $(x, y) = (\xi - x_s, \eta - y_s)$. The free-surface component G^F is given by the following double integral

$$4\pi^2 G^F = \lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} dk_1 \frac{A\mathrm{e}^{-i(k_1 x + k_2 y)}}{D - i\varepsilon\,\mathrm{sign}(k_1)} \tag{3}$$

where $A = \cosh k(\xi+1)\cosh k(z_s+1)/\cosh^2 k$ with $k = \sqrt{k_1^2+k_2^2}$, and $D = (k_1 F)^2 - k\tanh k$ with F the depth Froude number. Using polar Fourier variables (k, θ), we can write Eqn. (3) as

$$4\pi^2 G^F = \lim_{\varepsilon\to+0}\int_{-\pi}^{\pi} d\theta \int_0^{\infty} dk \frac{A\mathrm{e}^{-ik(x\cos\theta + y\sin\theta)}}{D/k - i\varepsilon\,\mathrm{sign}(\cos\theta)} \tag{4}$$

where $D = (kF\cos\theta)^2 - k\tanh k$. Interchange the order of integration and taking the limit $\varepsilon \to +0$, we have

$$4\pi^2 G^F = \int_0^{\infty} dk\ A \int_L d\theta \frac{\mathrm{e}^{-ik(x\cos\theta + y\sin\theta)}}{kF^2\cos^2\theta - \tanh k} \tag{5}$$

where path L, as shown in Figure 1, is dependent on the existence and location of the zeroes of the dispersion function D. Let's introduce the critical wavenumber k^* which corresponds to waves with phase velocity U. k^* is given implicitly by $\sqrt{\tanh k^*/k^*} = F$. For $F > 1$, k^* is defined to be zero since there are no waves with phase velocity U. When $k < k^*$, there are no poles in the integrand of (5), and path L is simply a straight line on the real axis from $-\pi$ to π. When $k > k^*$, there are four poles given by

$$\theta_1, \theta_2 = \pm\cos^{-1}(\sqrt{\tanh k/(kF^2)}) \qquad \text{and} \qquad \theta_3, \theta_4 = \pm\cos^{-1}(-\sqrt{\tanh k/(kF^2)}) \tag{6}$$

Path L now has indentations around these poles, and the location of the indentations in the upper or lower half of the complex plane is determined by the limit $\varepsilon \to +0$ and is shown in Figure 1.

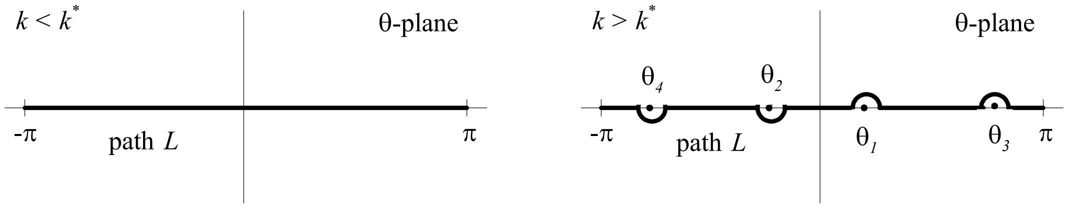

Figure 1. Definition of path L for $k < k^*$ and $k > k^*$

In the next section, we will obtain a single-integral representation for G^F by evaluating the integral Θ below using Cauchy' residue theorem.

$$\frac{4}{ikF^2}\Theta = \int_L d\theta \frac{e^{-ik(x\cos\theta + y\sin\theta)}}{kF^2\cos^2\theta - \tanh k} \tag{7}$$

2. Evaluation of integral Θ

The integral Θ can be evaluated using Cauchy's residue theorem. Using the transformation $\beta = e^{i\theta}$, we can rewrite Θ in terms of the new variable β as follows:

$$\Theta = \int_C f(\beta)\, d\beta = \int_C \frac{\beta e^{-\frac{ik}{2}[x(\beta+\beta^{-1}) - iy(\beta-\beta^{-1})]}}{\beta^4 + 2(1-2a^2)\beta^2 + 1} d\beta \tag{8}$$

where $a = \sqrt{\tanh k/(kF^2)}$ is the ratio of the phase velocity of waves with wavenumber k to the source's velocity. The path of integration C is now a unit circle centered at the origin of the β-plane and has indentations corresponding to those of path L as shown in Figure 2. The integrand f has five singularities. Four are simple poles associated with the zeroes of the denominator. The remaining singularity is at $\beta = 0$ and is an essential singularity. The four simple poles of f are given by:

$$\beta_1, \beta_2 = a \pm \sqrt{a^2 - 1} \qquad \text{and} \qquad \beta_3, \beta_4 = -a \pm \sqrt{a^2 - 1} \tag{9}$$

These poles are either all real or all complex depending on whether a is greater than or less than unity. For $F > 1$, the source's velocity is greater than the phase velocity of all waves, and a is less than unity for all k. All four simple poles are complex in this case. For $F < 1$, k^* is the wavenumber for waves with phase velocity equal to the source's velocity. Therefore, $a = 1$ when $k = k^*$. Also, $a > 1$ for $k < k^*$ and $a < 1$ for $k > k^*$. Thus, the poles are real when $k < k^*$ and complex when $k > k^*$ as illustrated in Figure 2. Note that since we define $k^* = 0$ for $F > 1$, the condition $k > k^*$ is always true, and Figure 2 is also valid for the supercritical case.

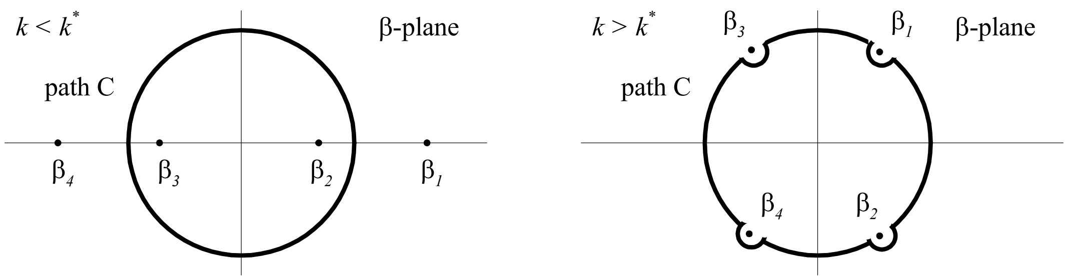

Figure 2. Definition of path C in the β-plane

To apply Cauchy's residue theorem to Eqn. (8), we need to obtain the residues of f for all singularities inside path C. We will first consider the residue at the essential singularity $\beta = 0$. The residues for the simple poles will be treated next for the two cases when $k < k^*$ and $k > k^*$. Let's introduce the polar coordinates (R,

ψ) where $R^2 = x^2 + y^2$, $x = R\cos\psi$, and $y = R\sin\psi$. With these variables, we can rewrite the terms involving x and y in Eqn. (8) as:

$$x(\beta + \beta^{-1}) - iy(\beta - \beta^{-1}) = R[\cos\psi(\beta + \beta^{-1}) - i\sin\psi(\beta - \beta^{-1})] = R(\beta\gamma^{-1} + \beta^{-1}\gamma) \tag{10}$$

where $\gamma = \mathrm{e}^{i\psi}$. Using Eqn. (10) and Eqn. 9.1.41 of Abramowitz & Stegun (1970), the exponential term of Eqn. (8) becomes

$$\mathrm{e}^{-\frac{ik}{2}[x(\beta+\beta^{-1})-iy(\beta-\beta^{-1})]} = \mathrm{e}^{\frac{kR}{2}[i^{-1}\gamma^{-1}\beta-(i^{-1}\gamma^{-1}\beta)^{-1}]} = \sum_{n=-\infty}^{\infty}(i\gamma)^{-n}\beta^n J_n(kR) \tag{11}$$

Furthermore, we can expand the denominator of f in terms of a Taylor series about $\beta = 0$

$$\frac{1}{\beta^4 + 2(1-2a^2)\beta^2 + 1} = \frac{1}{\beta_1^2 - \beta_1^{-2}}\sum_{m=0}^{\infty}\beta^{2m}\left[\beta_1^{2(m+1)} - \beta_1^{-2(m+1)}\right] \tag{12}$$

Substituting Eqns. (11) and (12) into (8), we obtain the following Laurent series expansion of f about $\beta = 0$

$$f = \frac{1}{\beta_1^2 - \beta_1^{-2}}\sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty}(i\gamma)^{-n}J_n(kR)\left[\beta_1^{2(m+1)} - \beta_1^{-2(m+1)}\right]\beta^{2m+n+1} \tag{13}$$

The residue of f at $\beta = 0$ is simply the coefficient of the β^{-1} - term of the Laurent series and is given by

$$\mathrm{res}\,(f,0) = \frac{1}{4a\sqrt{a^2-1}}\sum_{m=1}^{\infty}(-1)^m\gamma^{2m}(\beta_1^{2m} - \beta_1^{-2m})J_{2m}(kR) \tag{14}$$

Residues at simple poles for $k < k^*$

Since $k^* = 0$ for $F > 1$, the condition $k < k^*$ can only occurs for subcritical Froude numbers. The velocity ratio a in this case is greater than unity, and the simple poles in Eqn. (9) are real with β_2 and β_3 located inside path C as shown in Figure 2. These poles do not correspond to any zeroes of the dispersion function D, and, therefore, their residues seem to contribute only to the local effects. The residues of f at β_2 and β_3 are easily obtained, and their sum is given by

$$\mathrm{res}\,(f,\beta_2) + \mathrm{res}\,(f,\beta_3) = \frac{-\cos(xka)\cosh(yk\sqrt{a^2-1}) + i\sin(xka)\sinh(yk\sqrt{a^2-1})}{4a\sqrt{a^2-1}} \tag{15}$$

Applying Cauchy's residue theorem to Eqn. (8) and using the results in Eqns. (14) and (15), we obtain the following expression for the integral Θ valid for $k < k^*$

$$\Theta = \frac{\pi i}{2a\sqrt{a^2-1}}\left[-\cos(xka)\cosh(yk\sqrt{a^2-1}) + i\sin(xka)\sinh(yk\sqrt{a^2-1}) + \sum_{m=1}^{\infty}(-1)^m\gamma^{2m}(\beta_1^{2m} - \beta_1^{-2m})J_{2m}(kR)\right] \tag{16}$$

Residues at simple poles for $k > k^*$

The velocity ratio a is less than unity in this case, and all simple poles in Eqn. (9) are complex. These poles are the transforms of the four zeroes θ_1, θ_2, θ_3, and θ_4 of the dispersion function D. Only two of the poles, β_2 and β_4, lie inside path C, and their residues contribute to the far-field waves. The sum of the residues at β_2 and β_4 is given by

$$\mathrm{res}\,(f,\beta_2) + \mathrm{res}\,(f,\beta_4) = \frac{\mathrm{e}^{iky\sin\theta_1}\sin(kx\cos\theta_1)}{2\sin 2\theta_1} \tag{17}$$

Since $\beta_1 = \mathrm{e}^{i\theta_1}$, we have $a = \cos\theta_1$, and the residue at $\beta = 0$, as given in Eqn. (14), can be simplified to

$$\mathrm{res}\,(f,0) = \frac{1}{\sin 2\theta_1}\sum_{m=1}^{\infty}(-1)^m\gamma^{2m}\sin(2m\theta_1)J_{2m}(kR) \tag{18}$$

Using Eqns. (17) and (18), we obtain the following expression for the integral Θ valid for $k > k^*$

$$\Theta = \frac{\pi i}{\sin 2\theta_1}\left[\mathrm{e}^{iky\sin\theta_1}\sin(kx\cos\theta_1) + 2\sum_{m=1}^{\infty}(-1)^m\gamma^{2m}\sin(2m\theta_1)J_{2m}(kR)\right] \tag{19}$$

With Θ defined for all values of k, we can now rewrite the free-surface component G^F in Eqn. (5). Making use of the symmetry property of G^F about the axis $y = 0$, it's easy to show that the complex part of G^F vanishes. The final form of G^F valid for all Froude numbers is given by

$$2\pi G^F = \int_0^{k^*} dk \frac{A}{kF^2 a\sqrt{a^2-1}} \left[-\cos(xka)\cosh(yk\sqrt{a^2-1}) + \sum_{m=1}^{\infty} (-1)^m (\beta_1^{2m} - \beta_1^{-2m}) \cos(2m\psi) J_{2m}(kR) \right]$$
$$+ \int_{k^*}^{\infty} dk \frac{2A}{kF^2 \sin 2\theta_1} \left[\sin(kx\cos\theta_1)\cos(ky\sin\theta_1) + 2\sum_{m=1}^{\infty} (-1)^m \sin(2m\theta_1)\cos(2m\psi) J_{2m}(kR) \right] \quad (20)$$

3. Alternative expressions for the infinite series

Both infinite series in Eqn. (20) converge slowly when $kR >> 1$, but the value of k in the first series is limited by k^*, and in most applications the computation of this series does not present major difficulty. The second series, however, is harder to evaluate in its current form when kR is large since an excessive number of terms is needed. Therefore, for $kR >> 1$, we develop three alternative representations that are more amenable to numerical computation. The second infinite series in (20) can be expressed in terms of series of the form

$$S(h,\alpha) = 2\sum_{m=1}^{\infty} (-1)^m \sin(2m\alpha) J_{2m}(h) \quad (21)$$

where $h = kR$ and $\alpha = \theta_1 \pm \psi$. The first representation of $S(h,\alpha)$ is obtained by considering a second-order ordinary differential equation satisfied by $S(h,\alpha)$ and making use of the Fourier sine transform. It's written as

$$S(h,\alpha) = -\sin(h\cos\alpha) + \frac{\sin(2\alpha)}{\pi} \Im\left\{ \int_0^{\infty} dt \frac{(\cos h + i\sin h)\,\mathrm{e}^{-ht}}{(\sin^2\alpha - t^2 + 2it)\sqrt{t^2 - 2it}} \right\} \quad (22)$$

where $\Im\{\cdot\}$ denotes the imaginary part of the expression inside the brackets. Using the above integral representation, we can develop the following asymptotic expansion

$$S(h,\alpha) = -\sin(h\cos\alpha) + \frac{\cos\alpha}{\sqrt{\pi}} \sum_{n=0}^{\infty} a_n [\sin h + (-1)^n \cos h](h\sin^2\alpha)^{-n-1/2} \quad (23)$$

where the first 5 coefficients a_n of the series are given by

$$\begin{aligned} &a_0 = 1, && a_1 = 1 + \sin^2\alpha/8, \\ &a_2 = -3 + 3\sin^2\alpha/8 - 9\sin^4\alpha/128, && a_3 = -15 + 45\sin^2\alpha/8 + 15\sin^4\alpha/128 - 75\sin^6\alpha/1024, \\ &a_4 = 105 - 525\sin^2\alpha/8 + 315\sin^4\alpha/128 - 105\sin^6\alpha/1024 + 3675\sin^8\alpha/32768 \end{aligned} \quad (24)$$

The second representation in (23) is much better suited for numerical computation than the series in (21) when $h\sin^2\alpha >> 1$. However, it is not useful when $h\sin^2\alpha < 1$. To complement (23), we develop the following ascending series using the Taylor series of $\sin(2m\alpha)$ and the recurrence relation for the Bessel functions

$$S(h,\alpha) = h\sin\alpha \sum_{n=0}^{\infty} b_n (h\sin^2\alpha)^n \quad (25)$$

where the first 5 coefficients b_n are given by

$$\begin{aligned} &b_0 = -J_1(h), \quad b_1 = J_0(h)/3 - J_1(h)/(6h), \quad b_2 = J_1(h)/15 + J_0(h)/(10h) - 9J_1(h)/(120h^2), \\ &b_3 = -J_0(h)/105 + 9J_1(h)/(210h) + 9J_0(h)/(168h^2) - 15J_1(h)/(336h^3), \\ &b_4 = -J_1(h)/945 - J_0(h)/(126h) + 5J_1(h)/(168h^2) + 5J_0(h)/(144h^3) - 35J_1(h)/(1152h^4) \end{aligned} \quad (26)$$

Eqn. (25) can be used when $h\sin^2\alpha < 1$, but when $h\sin^2\alpha = \mathrm{O}(1)$, Eqn. (22) is more appropriate and can be evaluated using the algorithm (25.4.45) in [1] for infinite integrals with exponentially decreasing integrands.

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Question by : J.N. Newman

What is the limit of your new single integral when the depth tends to infinity ?

Author’s reply:

For the purpose of making depth tends to infinity, we may write the new single integral in another form by using ship's length as reference length instead of waterdepth. The limit of the modified single integral is then the Green function in deep water when the depth tends to infinity and the wavenumber in the integral is kept constant. Same as the classical integral representation, there is a non-uniformity about the wavenumber tends to zero. This peculiar property has been treated in Chen & Zhao (2001) by removing constants embedded in the formulations.