Wave Scattering in the Marginal Ice Zone

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1 Introduction

The Marginal Ice Zone is an interfacial region of ice floes which forms at the boundary of open water the continuous pack ice. The following photograph of the Marginal Ice Zone shows a typical situation.

A photograph of the Marginal Ice Zone taken from a ship in the Antractic.

The major way the open ocean interacts with the interior continuous ice is through wave induced breaking of the ice. It is this wave induced breaking which produces the Marginal Ice Zone. However wave action does not break up the continuous ice for an indefinite distance. Instead, the wave energy is dissipated by scattering from the ice floes which have formed at the ice edge. The Marginal Ice Zone is thus formed by wave induced breaking of the continuous ice and simultaneously shields the continuous ice from breaking. There are two aspects which need to be understood to model this process, the first is the wave induced breaking of the continuous ice, and the second is wave scattering by the Marginal Ice Zone. This paper presents a model for the latter process.

The main experimental studies of wave propagation in the Marginal Ice Zone, reported in Wadhams *et al.* (1986) and Wadhams *et al.* (1988), have shown the following features of wave propagation in the Marginal Ice Zone. There is strong exponential attenuation of energy, which decreases as the period increases. From a narrow directional spectrum at the ice edge the wave field broadens and becomes isotropic as it evolves with increasing distance into the Marginal Ice Zone.

The incoming waves from the open ocean are scattered by the presence of ice floes on the water surface. There are two length scales at which this scattering is occurring. The small scale is the length scale of an individual ice. In the Marginal Ice Zone this length scale is of the order of the wavelength. The individual floe motion is the major cause of the scattering of wave energy by the Marginal Ice Zone. The large length scale is the scale of many wavelengths. At this length scale the wave is no longer coherent and we must consider an equation for the propagation of energy. Of course the wave scattering at the large scale is dependent on the scattering at the small scale.

To model scattering at the short scale we need a model for the motion and scattering of an individual ice floe. This model must take into account the following features of "real" ice floes. Real ice floes are of irregular shape. Experiments have shown, and theory predicts, that ice floes tend to form with a size which is between a quarter and half a wavelength. The wavelengths of interest are hundreds of metres, floe thicknesses are almost always a metre or less. This means that ice floes have a small relative stiffness and can bend in compliance with the waves. This bending is observed in measurements of ice floe motions Squire (1983) and Squire & Martin (1980). This means that an ice floe must be modelled as flexible and of arbitrary shape. Such a model has only recently been developed by Meylan (2002).

The large scale equations are governed by scattering theory. There are two approaches to scattering theory, multiple scattering, and the transport equation (Ishimaru (1978)). Two models have been proposed for the large scale scattering in the Marginal Ice Zone, Masson & LeBlond (1989) used multiple scattering and Meylan *et al.* (1997) used the transport equation. However, neither Masson & LeBlond (1989) nor Meylan *et al.* (1997) used the correct model for the scattering kernel. In this paper we develop a model based on the transport equation, which corrects errors in the models presented by Masson & LeBlond (1989) and Meylan *et al.* (1997). Furthermore, we use a fully compliant ice floe model.

2 The Equation for Large Scale Wave Scattering in the Marginal Ice Zone.

We begin by deriving an equation for the large length, or time, scale propagation of wave energy through the Marginal Ice Zone. The large length scale is many wavelengths, the large time scale is many wave periods. Over these large scales the waves are essentially incoherent. Since the waves are incoherent at the large scale the equation is for energy not displacement. We will develop a scattering theory using the Boltzmann (transport) equation for the following reasons. The Boltzmann is a comparatively simple equation. It does not make sense to develop a theory which concerns itself with the last few percents of accuracy but which hinges on assumptions which themselves must destroy accuracy at this scale (or example the assumption of uniform floe size distribution).

We consider the surface of an infinitely deep ocean, which we represent in Cartesian coordinates by x and y. Wave energy is propagating across this surface in all directions so that at any point we must consider the energy travelling in each direction. We introduce an intensity function $I(x, y, t, \theta)$ which is the rate of flow of energy travelling in a given direction, per unit of surface, per unit of angle. In the absence of scatterers, if we assume that the waves continue to propagate in the same direction the intensity of energy satisfies the following equation,

$$\frac{1}{c_g}\frac{\partial}{\partial t}I(x,y,t,\theta) + \theta.\nabla I(x,y,t,\theta) = 0, \qquad [2.1]$$

(Phillips (1977)) where c_g is the velocity of propagation (the deep water group speed). The presence of the floes will modify this expression by scattering energy, i.e. changing the direction in which the energy is travelling. The equation of transport for the propagation of wave energy through a scattering medium is given in Howells (1960) as,

$$\frac{1}{c_g}\frac{\partial}{\partial t}I(x,y,t,\theta) + \theta.\nabla I(x,y,t,\theta) = -\beta(x,y,\theta)I \qquad [2.2]$$

$$+\int_0^{2\pi} S(x,y,\theta,\theta')I(x,y,t,\theta')d\theta', \qquad [2.3]$$

where β is the absorption coefficient and S is the scattering function, which we allow to vary with position but not with time. The absorption coefficient, $\beta(x, y, \theta)$, is the fraction of energy lost by scattering and dissipative processes (assumed linear) from a pencil of radiation in direction θ, per unit path length travelled in the medium. The scattering function $S(x, y, \theta, \theta')$ specifies the angular distribution of scattered energy in such a way that,

$$S(x,y,\theta,\theta')I(x,y,t,\theta')d\Omega dS d\Omega',$$

is the rate at which energy is scattered from a pencil of radiation of intensity $I(\mathbf{r}, t, \theta')$ at an angle $d\Omega'$ in direction θ', by a surface dS at position x, y, into an angle $d\Omega$ in direction θ.

To apply equation ([2.3]) to wave scattering in the Marginal Ice Zone we must estimate the absorption coefficient $\beta(x, y, \theta)$ and the scattering function $S(x, y, \theta, \theta')$. The absorption coefficient can be found from the scattering function if we assume that there is no dissipation of energy. Alternatively it must be estimated in some fashion, either from some theory or from experimental measurements. The latter method was used by Masson & LeBlond (1989). The scattering function is determined by calculating the scattering from a single ice floe.

3 Wave Scattering by a Single Ice Floe

To modelling wave scattering in the Marginal Ice Zone we must develop a model for wave scattering by a single ice floe. The floe will be considered to be surrounded by open water and subject wave field which consists of a plane wave of a single frequency. A floating thin plate of negligible submergence has been the standard model for an ice floe since Wadhams (1986) and it is described in detail in Squire *et al.* (1995). The solution to this problem has only recently been developed by Meylan (2002) and no other work on wave propagation in the Marginal Ice Zone has used this ice floe model.

We will determine the response of the ice floe to wave forcing of a single frequency. Therefore we write all variables as the real part of a complex function multipied by $e^{-i\sqrt{\alpha}t}$ where $\sqrt{\alpha}$ is the radian frequency. In the complex, non-dimensional, variables the equation of motion of the ice floe is

$$\beta\nabla^4 w + \alpha\gamma w = \sqrt{\alpha}\phi - w. \qquad [3.1]$$

The definition of β and γ and the non-dimensionalisation is described in detail in Meylan (2002).

3.1 Equations of Motion for the Water

We require the equation of motion for the water to solve equation ([3.1]). We begin with the non-dimensional equations of potential theory which describe linear surface gravity waves

$$\left.\begin{aligned} \nabla^2\phi &= 0, && -\infty < z < 0, \\ \frac{\partial\phi}{\partial z} &= 0, && z \to -\infty, \\ \frac{\partial\phi}{\partial z} &= -i\sqrt{\alpha}w, && z = 0, \ \mathbf{x} \in \Delta, \\ \frac{\partial\phi}{\partial z} - \alpha\phi &= p, && z = 0, \ \mathbf{x} \notin \Delta, \end{aligned}\right\} \qquad [3.2]$$

(Wehausen & Laitone (1960)). As before, w is the displacement of the floe and p is the pressure at the water surface. The vector $\mathbf{x} = (x, y)$ is a point on the water surface and Δ is the region of the water surface occupied by the floe. The boundary value problem ([3.2]) is subject to an incident wave which is imposed through the Sommerfeld radiation condition as $|\mathbf{x}| \to \infty$ (Wehausen & Laitone (1960)).

The standard solution method to the linear wave problem is to transform the boundary value problem into an integral equation using a Green function (John (1949, 1950)). Performing such a transformation, the boundary value problem ([3.2]) becomes

$$\phi(\mathbf{x}) = \phi^i(\mathbf{x}) + \iint_\Delta G_\alpha(\mathbf{x};\mathbf{y})\left(\alpha\phi(\mathbf{x}) + i\sqrt{\alpha}w(\mathbf{x})\right)dS_{\mathbf{y}}. \qquad [3.3]$$

The Green function G_α is

$$G_\alpha(\mathbf{x};\mathbf{y}) = -\frac{1}{2\pi|\mathbf{x}-\mathbf{y}|} + \frac{i\alpha}{2}J_0(\alpha|\mathbf{x}-\mathbf{y}|) + \frac{\alpha}{4}\left(\mathbf{H_0}(\alpha|\mathbf{x}-\mathbf{y}|) + Y_0(\alpha|\mathbf{x}-\mathbf{y}|)\right)$$

(Kim (1965)), where J_0 and Y_0 are respectively Bessel functions of the first and second kind of order zero, and $\mathbf{H_0}$ is the Struve function of order zero (Abramowitz & Stegun, 1964).

3.2 Solving for the Wave Induced Ice Floe Motion

To determine the ice floe motion we must solve equations ([3.1]) and ([3.3]) simultaneously. We do this by expanding the floe motion in the free modes of vibration of a thin plate. Since the operator ∇^4, subject to the free edge boundary conditions, is self adjoint a thin plate must possess a set of modes w_i which satisfy the free boundary conditions and the following eigenvalue equation

$$\nabla^4 w_i = \lambda_i w_i.$$

The modes which correspond to different eigenvalues λ_i are orthogonal and the eigenvalues are positive and real. While the plate will always have repeated eigenvalues, orthogonal modes can still be found and the modes can be normalized. We therefore assume that the modes are orthonormal, i.e.

$$\iint_{\Delta} w_i(\mathbf{Q})\, w_j(\mathbf{Q})\, dS_{\mathbf{Q}} = \delta_{ij}$$

where δ_{ij} is the Kronecker delta. The eigenvalues λ_i have the property that $\lambda_i \to \infty$ as $i \to \infty$ and we order the modes by increasing eigenvalue. These modes can be used to expand any function over the wetted surface of the ice floe Δ.

We expand the displacement of the floe in a finite number of modes N, i.e.

$$w(\mathbf{x}) = \sum_{i=1}^{N} c_i w_i(\mathbf{x}). \qquad [3.4]$$

From the linearity of ([3.3]) the potential can be written in the following form

$$\phi = \phi_0 + \sum_{i=1}^{N} c_i \phi_i \qquad [3.5]$$

where ϕ_0 and ϕ_i satisfy the integral equations

$$\phi_0(\mathbf{x}) = \phi^{\mathrm{In}}(\mathbf{x}) + \iint_{\Delta} \alpha G_\alpha(\mathbf{x};\mathbf{y})\phi(\mathbf{y}) dS_{\mathbf{y}} \qquad [3.6]$$

and

$$\phi_i(\mathbf{x}) = \iint_{\Delta} G_\alpha(\mathbf{x};\mathbf{y}) \left(\alpha\phi_i(\mathbf{x}) + i\sqrt{\alpha} w_i(\mathbf{y})\right) dS_{\mathbf{y}}. \qquad [3.7]$$

The potential ϕ_0 represents the potential due the incoming wave assuming that the displacement of the ice floe is zero. The potentials ϕ_i represent the potential which is generated by the plate vibrating with the ith mode in the absence of any input wave forcing.

We substitute equations ([3.4]) and ([3.5]) into equation ([3.1]) to obtain

$$\begin{aligned} \beta \sum_{i=1}^{N} \lambda_i c_i w_i - \alpha\gamma \sum_{i=1}^{N} c_i w_i & \qquad [3.8] \\ = \; i\sqrt{\alpha}\left(\phi_0 + \sum_{i=1}^{N} c_i\phi_i\right) - \sum_{i=1}^{N} c_i w_i. & \end{aligned}$$

To solve equation ([3.8]) we multiply by w_j and integrate over the plate (i.e. we take the inner product with respect to w_j) taking into account the orthogonality of the modes w_i, and obtain

$$\begin{aligned} & \beta\lambda_j c_j + (1-\alpha\gamma) c_j \qquad [3.9] \\ = & \iint_{\Delta} i\sqrt{\alpha}\left(\phi_0(\mathbf{Q}) + \sum_{i=1}^{N} c_i\phi_i(\mathbf{Q})\right) w_j(\mathbf{Q})\, dS_{\mathbf{Q}} \end{aligned}$$

which is a matrix equation in c_i.

We cannot solve equation ([3.9]) without determining the modes of vibration of the thin plate w_i (along with the associated eigenvalues λ_i) and solving the integral equations ([3.6]) and ([3.7]). We use the finite element method to determine the modes of vibration (Zienkiewicz & Taylor (1989)) and the integral equations ([3.6]) and ([3.7]) are solved by a constant panel method (Sarpkaya & Isaacson (1981)). The same set of nodes is used for the finite element method and to define the panels for the integral equation.

4 Deriving the Scattering Kernel in the Boltzmann Equation from Wave Scattering by a Single Floe

For our model of wave scattering in the MIZ we require the amount of wave energy the ice floe scatters. The scattered energy can be calculated from the Kochin function, $H(\tau)$, (Wehausen & Laitone (1960)) which is defined by the equation

$$H(\tau) = \int\int_{\Delta} (k\phi + i\omega W)\, e^{ik(x\cos\tau + y\sin\tau)} dS. \qquad [4.1]$$

The energy radiated by per unit angle per unit time, $E(\tau)$, is given by,

$$E(\tau) = \frac{\rho\omega^3}{8\pi g} |H(\pi+\tau)|^2. \qquad [4.2]$$

We must now express the scattering kernel in equation ([2.3]) $S(x,y,t,\theta,\theta')$ in terms of E. Obviously the spatial variation in S depends in our model on the ice floe geometry at the point (x,y).

We recall that the scattering function $S(x,y,t,\theta,\theta')$ specifies the angular distribution of scattered energy in such a way that,

$$S(x,y,\theta,\theta')I(x,y,t,\theta')d\Omega dS d\Omega',$$

is the rate at which energy is scattered from a pencil of radiation of intensity $I(\mathbf{r},t,\theta')$ at an angle $d\Omega'$ in direction θ', by a surface dS at position x,y, into an angle $d\Omega$ in direction θ. Therefore to find S from E we must divide E by the rate of energy which is passing under the ice floe. The rate of energy passing under the floe is given by the product of the wave energy density ($\frac{1}{2}\rho g A^2$, where A is the wave amplitude), the average area occupied by a floe (A_f/c, where A_f is the area of the floe and c is the floe concentration), and the wave group speed $(\omega/2k)$. This gives us the following expression for S,

$$\begin{aligned} S(\theta,\theta') &= \frac{4kcE(\theta-\theta')}{A_f\rho g A^2\omega} \qquad [4.3] \\ &= \frac{kc}{A_f A^2}\frac{\omega^2}{2\pi g^2}|H(\pi+\theta-\theta')|^2 \end{aligned}$$

There is an error in the expression for S in Meylan *et al.* (1997) because the wave phase speed not group speed was used. The expression for S used by Masson & LeBlond (1989), with errors corrected can also be shown to agree with this expression in the low concentration limit.

5 Numerical Solution of the Transport Equation

Our aim is to predict the wave propagation in the Marginal Ice Zone. We have determined the scattering function using equation ([4.3]). To predict the wave propagation we must solve the linear Boltzmann equation ([2.3]). This equation can only be solved by making further assumptions. Masson & LeBlond (1989) assumed there was only variation in time and Meylan *et al.* (1997) assumed only variation in one spatial direction. We will present both of these solution methods.

To simplify equation [2.3] we assume that the solution is only a function of the x spatial coordinate and time, i.e.

there is no y dependence on the solution. We also consider a uniform Marginal Ice Zone so that the scattering function, S, is a function only of θ and θ' and β is a constant. The only variation we allow spatially is that the Marginal Ice Zone occupies the region $x > b$, i.e. the ice edge is at $x = b$. This will allow us to consider a wave spectrum which enters the Marginal Ice Zone from the open ocean. Under these assumptions equation [2.3] becomes,

$$\frac{1}{c_g}\frac{\partial I}{\partial t}+\cos\theta\frac{\partial I}{\partial x} = \begin{cases} -\beta I + \int_0^{2\pi} S(\theta-\theta')I(\theta')d\theta', & x > b, \\ 0, & x < b. \end{cases} \qquad [5.1]$$

The easiest way to determine β is to assume conservation of energy so that

$$\beta = \int_0^{2\pi} S(0,\theta')d\theta', \qquad [5.2]$$

however empirical damping could be included here.

To solve equation ([5.1]) we convert the problem to a matrix equation by introducing a discretisation in angle. We use a discrete ordinate method (Case & Zweifel (1967)) and represent the angular coordinate by a discrete set of n angles, $\theta_j = 2\pi j/n,\ 0 \le j \le n-1$, evenly spaced between 0 and 2π. This approximation converts equation ([5.1]) to the following equation,

$$\frac{1}{c_g}\frac{\partial \vec{I}}{\partial t} - \frac{\partial}{\partial x}\mathbf{D}\vec{I} = \begin{cases} -\beta I + \mathbf{S}\vec{I}, & x > b, \\ 0, & x < b. \end{cases} \qquad [5.3]$$

In equation ([5.3]) the intensity $\vec{I}$ is now a vector of values at the angle θ_j and the elements of the matrices $\mathbf{D}$ and $\mathbf{S}$ are given by,

$$d_{ij} = \begin{cases} -\cos\theta_i, & i = j, \\ 0, & i \neq j, \end{cases} \qquad [5.4]$$

and

$$s_{ij} = \begin{cases} -\beta + S(\theta_i - \theta_i)\dfrac{2\pi}{n}, & i = j, \\ S(\theta_i - \theta_j)\dfrac{2\pi}{n}, & i \neq j. \end{cases} \qquad [5.5]$$

We can solve equation ([5.3]) only in the stationary (no time dependence) or isotropic (no spatial dependence) case. In the stationary case equation ([5.3]) reduces to, setting the ice edge to $b = 0$,

$$-\frac{\partial}{\partial x}\mathbf{D}\vec{I} = -\beta\vec{I} + \mathbf{S}\vec{I}, \quad x > 0. \qquad [5.6]$$

For the isotropic case equation ([5.3]) reduces to, setting the ice edge to $b = -\infty$,

$$\frac{1}{c_g}\frac{\partial \vec{I}}{\partial t} = -\beta I + \mathbf{S}\vec{I}, \quad t > 0. \qquad [5.7]$$

Equations ([5.6]) and ([5.7]) can be solved by straightforward matrix methods (Ishimaru (1978)). Equation ([5.6]) requires as a boundary condition the wave spectrum at the ice edge, $x = 0$, and equation ([5.7]) requires as an initial condition the wave spectrum at $t = 0$. Due to the present constraint of space no numerical solutions are presented here.

6 Summary

A model for wave scattering in the Marginal Ice Zone has been presented which is based on the Boltzmann equation and an arbitrary geometry flexible ice floe model. We show how the ice floe model can be solved to give the scattering kernel in the Boltzmann equation and how the Boltzmann equation can be solved under futher assumptions.

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Question by : M. Tulin
I really enjoyed your talk.
How uniform is the ice thickness in the marginal ice zone?

Author's reply:
My understanding is that the ice is approximately uniform especially over length scales of the wavelength.

Comment by : M. Kashiwagi
The solution method adopted in this paper is essentially the same as the pressure distribution method which has been developed by me, although the modal functions for the elastic deflection are different.

Author's reply:
The solution methods are very similar. The major difference is that I use the free modes of vibration of an elastic plate calculated using the FEM while your method uses the modes of a beam.

Question by : I. Sturova
Did you solve the diffraction problem for any shape of ice floe and finite depth of fluid?

Author's reply:
Yes! The solution is described in Meylan (2002) referred to in my abstract.