Scattering of flexural-gravity waves by multiple cracks in ice sheets floating on water of finite depth

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1. Introduction

In this paper we consider the reflection of oblique flexural-gravity waves propagating along an ice-sheet by an arbitrary number of infinitely-long narrow parallel cracks of aribtrary spacing. The ice-sheet floats on water of constant finite depth and is modelled as an elastic plate whose edges are free to move. The motion of the plate and the fluid it rests upon is considered within the framework of linearised theory.

The simplest problem in this category involves a single crack in the ice sheet and this problem has recently been studied by the authors (Evans & Porter 2003). It was shown that the velocity potential and hence the reflection and transmission coefficients are given explicitly. In particular, it was shown that certain quantities relating to the jumps in displacement and gradient across the crack play a key rôle in determining the solution. Two methods of solution were used giving the same explicit solutions in terms of elementary infinite sums. One method utilises the appropriate Greens function and the other is based on the use of non-orthogonal eigenfunctions in the depth variable. In infinite depth, a similar approach has been used by Squire & Dixon (2000) and Williams & Squire (2002).

In this paper we extend the problem considered by Evans & Porter (2003) to one involving an arbitrary number of narrow parallel cracks of arbitrary spacing and approach the problem using non-orthogonal eigenfunctions. Thus we are able to demonstrate that the solution for N cracks may be expressed simply and without approximation in terms of the solution of a $2N \times 2N$ system of equations. As in the single crack problem, it is the quantities ΔQ_j and ΔP_j for $j = 1, \ldots, N$ relating to jumps in displacement and gradient across each of the cracks that play the key rôle in the solution process.

A related problem involves the scattering of waves by a *semi-infinite* periodic arrangement of cracks. However one cannot naïvely take the limit as $N \to \infty$ in the formulation described above since (i) the radiation condition at infinity is inconsistent and (ii) any truncation to a large but finite value of N (as would be needed by numerical computations for example) would lead to interference effects in reflection and transmission coefficients due to the two end cracks. Instead the original formulation can be modified by using information from a related problem involving an infinite periodic array of cracks (the so-called Bloch problem). This leads to another infinite system of equations which does not suffer problems with truncation since the condition at infinity has been incorporated explicitly into the formulation.

2. Formulation for scattering by multiple cracks

Cartesian coordinates are chosen with y directed vertically upwards. Fluid is bounded below by a flat rigid bottom on $y = -h$, $-\infty < x, z < \infty$. An infinite elastic plate of small thickness, d, floats on the surface of the fluid which has its mean position on $y = 0$. The plate is divided into N infinite strips by narrow parallel straight-line cracks occupying $x = a_j$, for $-\infty < z < \infty$, where $a_j \le x \le a_{j+1}$, $j = 1, \ldots, N$. See figure 1.

We seek a velocity potential $\Phi(x, y, z, t)$ in the region $-h < y < 0$, $-\infty < x, z < \infty$ and write $\Phi(x, y, z, t) = \Re\{\phi(x, y)e^{ilz - i\omega t}\}$ where ω is the radian frequency of motion and l is an assumed wavenumber in the z coordinate. It follows that the reduced velocity potential $\phi(x, y)$ satisfies

$$(\nabla^2 - l^2)\phi = 0, \quad -h < y < 0, \ -\infty < x < \infty, \tag{1}$$

$$\phi_y = 0, \quad \text{on } y = -h, \ -\infty < x < \infty, \tag{2}$$

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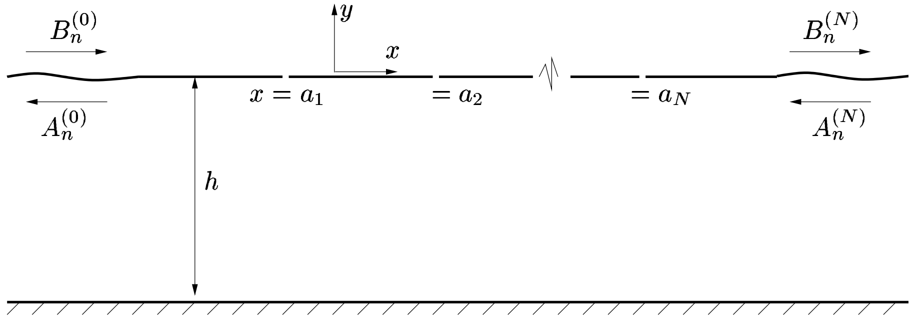

Figure 1: A general configuration of cracks

and

$$[\beta(\partial_{xx} - l^2)^2 + (1-\delta)]\phi_y - \kappa\phi = 0, \quad \text{on } y = 0,\ x \neq a_j,\ j = 1, 2, \ldots, N. \tag{3}$$

In (3), $\kappa = \omega^2/g$ where g is gravity, $\beta = D/(\rho_w g)$ where D is the flexural rigidity of the sheet and ρ_w is the density of water and $\delta = (\rho_i/\rho_w)\kappa d$ where ρ_i is the density of ice.

The method of solution involves expanding $\phi(x, y)$ in terms of eigenfunction expansions beneath each separate sheet and matching across common boundaries. Thus we write

$$\phi^{(r)}(x, y) = \sum_{n=-2}^{\infty} \{A_n^{(r)} \mathrm{e}^{-ik_n(x-a_{r+1})} + B_n^{(r)} \mathrm{e}^{ik_n(x-a_r)}\} Y_n(y) \tag{4}$$

for $a_r \leq x \leq a_{r+1}$ and for $r = 1, 2, \ldots, N-1$. In the expansion above we have defined $Y_n(y) = \cosh \gamma_n(y + h)$ which are non-orthogonal eigenfunctions but which satisfy the relation

$$\int_{-h}^{0} Y_n(y) Y_m(y) dy = C_n \delta_{nm} - \beta(\gamma_m^2 + \gamma_n^2) Y_m'(0) Y_n'(0) \tag{5}$$

where $C_n = \frac{1}{2}\{h + \kappa^{-1}(5\beta\gamma_n^4 + 1 - \delta)[Y_n'(0)/\gamma_n]^2\}$. Also, $\gamma_n = (k_n^2 + l^2)^{\frac{1}{2}}$ satisfy a dispersion relation

$$(\beta\gamma_n^4 + 1 - \delta)\gamma_n \tanh \gamma_n h - \kappa = 0 \tag{6}$$

governing possible values of k_n. It can be shown that this has a pair of real roots $\pm\gamma_0$ with corresponding roots $\pm k_0$ which describe progressive waves, provided $\gamma_0 > l$. In addition there is a sequence of pure imaginary roots $\pm k_n$ with $n = 1, 2, \ldots$ and four complex roots $\pm k_{-1}$ and $\pm k_{-2}$ symmetric about the real and imaginary axes. Let the roots in the first and second quadrants be k_{-1} and k_{-2}.

For $r = 0$ and $r = N$ the representation in (4) is still valid provided we define $a_0 = 0$ and $a_{N+1} = 0$ and provided the radiation condition is satisfied. Thus for an incident wave from $x = -\infty$ we have $B_n^{(0)} = \delta_{n0}$, $A_n^{(N)} = 0$, and the reflection and transmission coefficients are given by

$$R_N^- = A_0^{(0)} \mathrm{e}^{ik_0 a_1}, \qquad T_N^- = B_0^{(N)} \mathrm{e}^{-ik_0 a_N} \tag{7}$$

whilst for an incident wave from $x = +\infty$ we have $B_n^{(0)} = 0$, $A_n^{(N)} = \delta_{n0}$, and the reflection and transmission coefficients are given by

$$R_N^+ = B_0^{(N)} \mathrm{e}^{-ik_0 a_N}, \qquad T_N^+ = A_0^{(0)} \mathrm{e}^{ik_0 a_1}. \tag{8}$$

At the free edges of the strips, we must impose conditions of zero bending moments and zero shear stresses, which we express as

$$\mathcal{B}\phi^{(r)} \to 0,\ \mathcal{S}\phi^{(r)} \to 0, \quad \text{as } x \to a_r^+ \qquad \text{and} \qquad \mathcal{B}\phi^{(r-1)} \to 0,\ \mathcal{S}\phi^{(r-1)} \to 0, \quad \text{as } x \to a_r^- \tag{9}$$

for $r = 1, 2, \ldots, N$, where we have defined the operators $\mathcal{B}$ and $\mathcal{S}$ by

$$(\mathcal{B}u)(x) \equiv [(\partial_{xx} - \nu l^2) u_y]_{y=0} \quad \text{and} \quad (\mathcal{S}u)(x) \equiv [(\partial_{xxx} - \nu_1 l^2 \partial_x) u_y]_{y=0} \tag{10}$$

and where $\nu_1 = 2 - \nu$ with ν Poisson's ratio.

Quantities of particular significance associated with the edges of each of the strips are

$$\Delta P_r = \lim_{x \to a_r} [\phi_{xy}^{(r)} - \phi_{xy}^{(r-1)}]_{y=0}, \qquad \text{and} \qquad \Delta Q_r = \lim_{x \to a_r} [\phi_y^{(r)} - \phi_y^{(r-1)}]_{y=0} \tag{11}$$

representing, respectively, the jumps in slopes and elevation at the edges of each of the sheets.

After an algebraically complicated procedure of matching potentials and using various identities resulting from the edge conditions we find that the unknown coefficients satisfy

$$A_n^{(r)} \mathrm{e}^{ik_n b_r} - A_n^{(r-1)} = E_n^{(r)} = \beta(k_n g_n' \Delta Q_r + i g_n \Delta P_r)/2k_n C_n \tag{12}$$

and

$$B_n^{(r)} - B_n^{(r-1)} \mathrm{e}^{ik_n b_{r-1}} = F_n^{(r)} = \beta(k_n g_n' \Delta Q_r - i g_n \Delta P_r)/2k_n C_n \tag{13}$$

where $b_r = a_{r+1} - a_r$ and $g_n = (k_n^2 + \nu l^2) Y_n'(0)$, $g_n' = (k_n^2 + \nu_1 l^2) Y_n'(0)$. These are the key equations that provide the platform for the scattering by finite arrays of cracks, semi-infinite periodic arrays and the Bloch problem for infinite periodic arrays referred to in the Introduction.

For an incident wave from $x = \pm\infty$ we find, after applying the edge conditions, that

$$\left.\begin{aligned} 2g_0 \mathrm{e}^{\mp ik_0 a_r} &= \beta \sum_{j=1}^{N} \left(\Delta Q_j \, s_{jr}^{(0)} + i \Delta P_j \, s_{jr}^{(1)} \right) \\ \pm 2k_0 g_0' \mathrm{e}^{\mp ik_0 a_r} &= \beta \sum_{j=1}^{N} \left(\Delta Q_j \, s_{jr}^{(2)} + i \Delta P_j \, s_{jr}^{(0)} \right) \end{aligned} \right\} \qquad r = 1, 2, \ldots, N \tag{14}$$

where the unknowns ΔQ_j, ΔP_j, $j = 1, \ldots, N$ are to be solved for. In (14) we have defined

$$s_{jr}^{(0)} = t_{j-r} \sum_{n=-2}^{\infty} \frac{g_n g_n'}{C_n} \mathrm{e}^{ik_n |d_{jr}|}, \qquad s_{jr}^{(1)} = \sum_{n=-2}^{\infty} \frac{g_n^2}{k_n C_n} \mathrm{e}^{ik_n |d_{jr}|}, \qquad s_{jr}^{(2)} = \sum_{n=-2}^{\infty} \frac{k_n g_n'^2}{C_n} \mathrm{e}^{ik_n |d_{jr}|} \tag{15}$$

with $t_j = 1$ if $j \geq 1$ and $t_j = -1$ if $j \leq 0$ and $d_{jr} = a_j - a_r$. It then follows that for an incident wave from $x = -\infty$, for example,

$$R_N^- = -\sum_{j=1}^{N} E_0^{(j)} \mathrm{e}^{ik_0 a_j}, \qquad T_N^- = 1 + \sum_{j=1}^{N} F_0^{(j)} \mathrm{e}^{-ik_0 a_j} \tag{16}$$

with similar expressions for R_N^+ and T_N^+. In the case of a single crack ($N = 1$) with $a_1 = 0$, it is straightforward to show that the expressions for the reflection and transmission coefficients are precisely those given in Evans & Porter (2003).

3. The Bloch problem for an infinite periodic array of cracks and scattering by a semi-infinite periodic array

If the array of cracks is periodic with $b_r = b$, then we may make a Floquet or Bloch assumption that the potential satisfies

$$\phi^{(r-1)}(x, y) = \mu \phi^{(r)}(x, y), \qquad -\infty < r < \infty \tag{17}$$

where μ is a complex number which is to be regarded as the eigenvalue in this problem: if μ lies on the unit circle in the complex plane then the frequency ω lies in a *passing band* and waves propagate through the infinite array without decay; if $|\mu| \neq 1$ then ω lies in a *stopping band* and waves cannot pass through an infinite array.

The assumed form of the solution in (17) allows us to consider just one cell of width b containing a crack and it immediately follows from (4) that $A_n^{(r-1)} = \mu A_n^{(r)}$ and $B_n^{(r-1)} = \mu B_n^{(r)}$. Using this in (12)

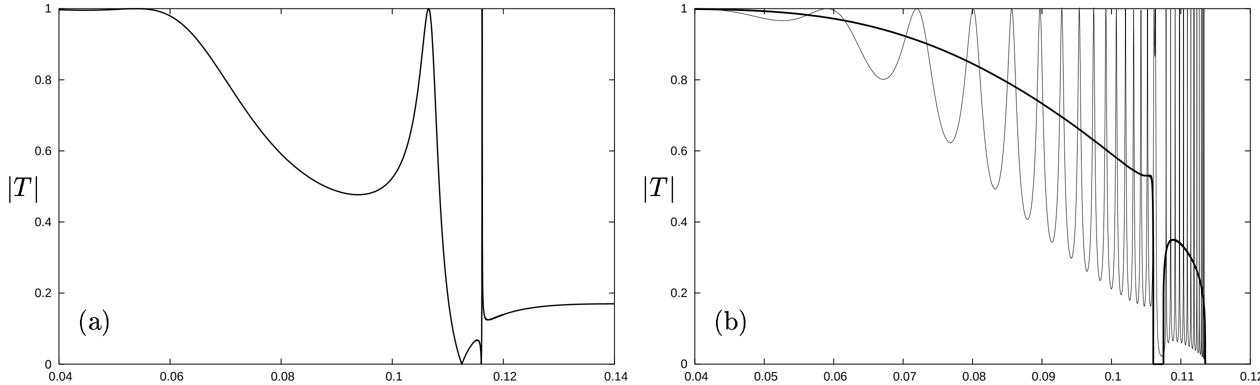

Figure 2: Transmission coefficient, $|T|$, against $k_0 d$ in the case of (a) two cracks with $b/d = \frac{1}{2}$ and (b) $N = 16$ cracks with $b/d = 2$ (thin line) and for the corresponding semi-infinite array (thick line).

and (13) results, once the edge conditions have been applied, in the following relation to be satisfied for the Bloch wave solution

$$s_{11}^2 - s_{12}s_{21} = 0 \tag{18}$$

where

$$s_{11} = \sum_{n=-2}^{\infty} \frac{g_n g_n'}{C_n}(v_n^- + v_n^+), \qquad s_{12} = \sum_{n=-2}^{\infty} \frac{g_n^2}{k_n C_n}(v_n^- - v_n^+), \qquad s_{21} = \sum_{n=-2}^{\infty} \frac{k_n g_n'^2}{C_n}(v_n^- - v_n^+) \tag{19}$$

with $v_n^\pm = 1/(1 - \mu \mathrm{e}^{\pm i k_n b})$.

For any given frequency of wave, we can use (18) to find μ for the infinite array of cracks of period b. We now consider the case where $a_j = jb$, $j = 1, 2, \ldots$ corresponding to an ice sheet containing a semi-infinite number of cracks and allow a wave to be incident from $x = -\infty$. Then as $x \to \infty$ we expect the solution to behave asymptotically according to (17) since far into the array of cracks the waves start to 'feel' as though they are part of an infinite array.

Thus we find it useful to define the modified coefficients $\widetilde{A}_n^{(r)} = A_n^{(r-1)} - \mu A_n^{(r)}$, and $\widetilde{B}_n^{(r)} = B_n^{(r-1)} - \mu B_n^{(r)}$ with similar definitions for $\widetilde{\Delta Q}_r$, $\widetilde{\Delta P}_r$, $\widetilde{E}_n^{(r)}$ and $\widetilde{F}_n^{(r)}$ where the particular μ for the corresponding infinite array is in use. It follows that all new coefficients vanish as $r \to \infty$ and we can either start with (12), (13) or go straight to (14) to derive a new infinite system of equations for the coefficients $\widetilde{\Delta Q}_j$ and $\widetilde{\Delta P}_j$, $j = 2, 3, \ldots$ plus ΔQ_1 and ΔP_1. In much the same way as already described for the scattering by a finite number of cracks, the reflection coefficient for the semi-infinite periodic array, R_∞^- can then be found in terms of a sum involving $\widetilde{\Delta Q}_j$, $\widetilde{\Delta P}_j$, ΔQ_1 and ΔP_1.

4. Results

A selection of results will be presented at the workshop covering a broad range of different configurations including scattering by single cracks, edge waves along cracks and non-uniqueness for multiple cracks. Here we only present two sets of results. In figure 2(a) the transmission coefficient is plotted against the incident wavenumber $k_0 d$ for a pair of cracks separated by $b/d = \frac{1}{2}$ for realistic ice parameters. Note the existence of zeros of transmission in this case and the spikey behaviour in the transmission coefficient which is associated with the close spacing of the cracks. In figure 2(b) we have plotted $|T|$ against $k_0 d$ again, but for an array of $N = 16$ cracks and compare those results with the transmission coefficient for a semi-infinite array of cracks $|T_\infty^-| = (1 - |R_\infty^-|^2)^{1/2}$.

5. References

Evans, D. V. & Porter, R. 2003 Wave scattering by narrow cracks in ice sheets on water of finite depth. *To appear in J. Fluid Mech.*

Squire, V.A. & Dixon A.W., 2000, An analytic model for wave propagation across a crack in an ice sheet. *Int. J. Offshore & Polar Eng.* , **10**, 173–176.

Williams, T.D. & Squire, V.A., 2002, Wave propagation across an oblique crack in an ice sheet. *Int. J. Offshore & Polar Eng.* , **12**(3), 157–162.

Question by : M. McIver

Can you make the eigenfunctions orthogonal if you use a weight function different from w(y)=1 ?

Author's reply:

I have not tried to do this as the method appears to work with w(y)=1.
