Interaction between water waves and elastic plates: Using the residue calculus technique

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1 Introduction

This paper reports a new method of computing the coefficients of the modal expansion of the velocity potential for the so-called linear hydroelasticity problem. Two examples of the hydroelasticity problems are studied, semi-infinite plate and finite-gap cases. The finite-gap depicts the situation when the ocean surface is covered with a thin elastic plate and the plate has a gap of a constant width. The mathematical model in this paper, a thin elastic plate coupled with an inviscid, incompressible fluid, is often used to describe sea-ice sheets and floating runways. The model can describe the dynamics of these objects for the wavelengths and wave-magnitudes typically found in the ocean. This article reports mainly the technical side of the method of solution and does not dwell on the geophysical and offshore engineering aspects of the problem.

Boundary-value problems of this kind, concerning a body of fluid with discontinuous boundary conditions, are usually numerically solved using the so-called mode-matching technique (Fox and Squire 1994; Lawrie and Abrahams 1999). The mode-matching technique uses the fact that the velocity potential of the fluid can be expressed as a modal expansion over the roots of the dispersion equation for the boundary conditions, e.g., elastic plates or open water. The modal expansion for the regions are then matched at the discontinuity. A system of equations for the coefficients of the modes is obtained as the result. The method of solution described in this paper, the Residue Calculus Technique (RCT) uses a complex function, which is constructed in such a way that the coefficients of the modes correspond to the residues at the function's poles. The closed form solution for the wave propagation across an infinite crack in an ice sheet has been reported in (Williams and Squire 2002) (using Green's function for an elastic plate) and in (Chung and Fox 2002) (using the Wiener-Hopf technique based (Evans and Davies 1968)).

The method here closely follows the method described in (Linton 2001) (rigid plate), particularly the introduction of the finite-length correction terms. The formula is based on the solution for the semi-infinite plate problem. The interaction between the ocean waves and the semi-infinite plate is solved in (Linton and Chung 2002) using the RCT. A difficulty arises when the plate is elastic, because there are two modes with complex wavenumbers (neither the real nor the imaginary part is zero). This is in contrast to the case of a

rigid plate, where the vertical modes are $\cos \pi n \left(z + H\right)/H$, $n = 0, 1, 2, ...$ under the rigid plate. In order for the coefficients to be determine completely, the boundary conditions to be applied at the plate ends called the edge conditions, are required. In (Linton and Chung 2002) the edge conditions were accommodated by introducing two arbitrary constants in the associated complex function. To the authors' knowledge, this method of incorporating the edge conditions to the RCT has not been reported before.

2 Method of solution

2.1 Governing equations

Consider a water way between two semi-infinite elastic plates. A plane wave is obliquely incident from infinity at an angle θ (to the x-axis). which is represented by $\exp\left[i\,\beta\left(x + L\right) + i\,ly\right]$. Assuming that the incident wave is sinusoidal in time, $\exp\left(-i\,\omega t\right)$, the y and t dependent part of the solution can be removed, because of the linearity of the system. Thus the boundary problem becomes a two dimensional problem in the (x, z) plane, i.e., the velocity potential, can be written as

$$\Psi\left(x, y, z, t\right) = \text{Re}\left[\Phi\left(x, z\right) e^{i\left(ly - \omega t\right)}\right]$$

where $\Phi\left(x, z\right)$ is a complex valued function.

The width of the water way is $2L$ and the flexural rigidity of the surrounding plates are D. The wave number along the y-axis, l, is given by the incident angle to the x-axis, θ, i.e., $l = \beta \sin \theta$ where $i\beta$ is the imaginary wave number corresponding to the mode propagating to infinity.

The system can be non-dimensionalized using length scale and time scales denoted by l_c and t_c, respectively. They are defined as

$$l_\text{c} = \left(\frac{D}{\rho g}\right)^{1/4}, \; t_\text{c} = \sqrt{\frac{l_\text{c}}{g}}$$

where D is the flexural rigidity, ρ is the mass density of the water, and g is the acceleration due to gravity. The flexural rigidity is computed using the formula, $D = Eh^3/\left(12\left(1 - \nu^2\right)\right)$, where E is the constant effective Young's modulus, h is the thickness of the plate, and ν is Poisson's ratio (set to be 0.3). The non-dimensionalization regime for sea-ice dynamic problem using the so-called characteristic length and time is reported in (Fox 2000).

The resulting non-dimensional differential equations for $\Phi_z\left(x, 0\right)$, z-derivative at the surface, and $\Phi\left(x, z\right)$, the (x, z)-

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part of the velocity potential are (see (Chung and Fox 2002))

$$\left(\partial_x^4 + 1 - \delta\right)\Phi_z - \omega^2\Phi = 0 \quad \text{for} \quad z = 0, |x| > L,$$
$$\left(\nabla_{x,z}^2 - l^2\right)\Phi = 0 \quad \text{for} \quad -H < z < 0,$$
$$\Phi_z = 0 \quad \text{for} \quad z = -H,$$
$$\Phi_z - \omega^2\Phi = 0 \quad \text{for} \quad z = 0, |x| < L.$$

For simplicity the inertial term is set as $\delta = m\omega^2$, where m is the non-dimensional mass density per unit area of the plate (normalized by ρl_c). The plate covered region for the semi-infinite plate case is $x > 0$, and the region $x < 0$ is free surface.

2.2 Method of solution

For simplicity the potential is decomposed into antisymmetric and symmetric potentials denoted by Φ^- and Φ^+, respectively, i.e., the solutions $\Phi^\pm$ satisfy the boundary conditions $\Phi_x^+ = 0$, $\Phi^- = 0$ on $x = 0$.

Define orthogonal sets of functions of z in the free surface region and plate covered region,

$$\phi_n(z) = N_n^{-1}\cos k_n(z+h),$$
$$N_n^2 = \tfrac{1}{2}h\left(1 + \frac{\sin 2k_nh}{2k_nh}\right), \quad \text{for } n \geq 0$$

$$\psi_n(z) = M_n^{-1}\cos\kappa_n(z+h),$$
$$M_n^2 = \tfrac{1}{2}\left(h\omega^2 - \left(5\kappa_n^4 + 1 - \delta\right)\sin^2\kappa_nh\right) \quad \text{for } n \geq -2$$

where k_n and κ_n are the roots of the dispersion equation

$$\omega^2 + k_n\tan k_nh = 0,$$
$$\omega^2 + \left(\kappa_n^4 + 1 - m\omega^2\right)\kappa_n\tan\kappa_nh = 0,$$

respectively. The orthogonality relation for free-surface region is

$$\int_{-h}^{0}\phi_m(z)\phi_n(z)\,dz = \delta_{mn}.$$

The subscript n for ψ includes -2 and -1 which are complex roots of the dispersion equation and that $\kappa_{-2} = \kappa_{-1}^*$ (* denotes the complex conjugate). The normalizing factors M_n are chosen so that

$$\omega^2\int_{-h}^{0}\psi_m(z)\psi_n(z)\,dz = \delta_{mn} + \left(\kappa_m^2 + \kappa_n^2\right)\psi_m'(0)\psi_n'(0) \tag{2.1}$$

where the prime indicates the z-derivative.

The modal expansion of the potentials are

$$\Phi^\pm(x,z) = \left(e^{-\beta_0(x+L)} + R^\pm e^{\beta_0(x+L)}\right)\psi_0(y)$$
$$+ \sum_{n=-2,n\neq0}^{\infty}b_n^\pm e^{\beta_n(x+L)}\psi_n(z), \quad x < -L,$$

$$\Phi^\pm(x,z) = \sum_{n=0}^{\infty}a_n^\pm\left(e^{\alpha_nx} \pm e^{-\alpha_nx}\right)\phi_n(z), \quad -L < x < 0,$$

where $\alpha_n = \sqrt{k_n^2 + l^2}$ and $\beta_n = \sqrt{\kappa_n^2 + l^2}$. Note that the pure imaginary wave numbers are $\alpha_0 = -i\alpha$ and $\beta_0 = -i\beta$ for the positive real numbers α and β. The reflection and the transmission coefficients can be obtained from $R = R^+ + R^-$, $T = R^+ - R^-$.

From the continuity conditions for $\Phi^\pm$ and $\Phi_x^\pm$ at $x = -L$, the equations for $a_n^\pm$ and $b_n^\pm$ after applying the orthogonality relation for ϕ_n are

$$2c_{0m} + \sum_{n=-2}^{\infty}b_n^\pm c_{nm} = \pm a_m^\pm\left(e^{\alpha_m L} \pm e^{-\alpha_m L}\right),$$
$$\sum_{n=-2}^{\infty}b_n^\pm\beta_n c_{nm} = \mp a_m^\pm\alpha_m\left(e^{\alpha_m L} \mp e^{-\alpha_m L}\right). \tag{2.2}$$

$$c_{nm} = \frac{A_mB_n}{\beta_n^2 - \alpha_m^2}, \quad A_m = \frac{\cos k_mh}{N_m}, \quad B_n = \frac{\kappa_n\sin\kappa_nh + \omega^2\cos\kappa_nh}{M_n}.$$

Notice that A_m and B_n are $O(1)$ as $n \to \infty$. Eliminating $a_m^\pm$ from the two equations then gives

$$\sum_{n=-2}^{\infty}V_n^\pm\left(\frac{1}{\beta_n - \alpha_m} \pm \frac{e^{-2\alpha_mL}}{\beta_n + \alpha_m}\right) = \frac{1}{\beta_0 + \alpha_m} \pm \frac{e^{-2\alpha_mL}}{\beta_0 - \alpha_m} \tag{2.3}$$

where $V_0^\pm = R^\pm$ and the coefficients $V_n^\pm = b_n^\pm B_n/B_0$ are the unknowns which will be determined using the residue calculus.

2.3 RCT

Define a function $f^\pm(\zeta)$, $\zeta \in \mathbb{C}$,

$$f^\pm(\zeta) = \frac{G^\pm g^\pm(\zeta)}{(\zeta + \beta_0)(\zeta - \beta_{-2})(\zeta - \beta_{-1})}\prod_{n=0}^{\infty}\frac{1 - \zeta/\alpha_n}{1 - \zeta/\beta_n},$$
$$g^\pm(\zeta) = \zeta^2 + \gamma_1^\pm\zeta + \gamma_2^\pm + \sum_{n=0}^{\infty}\frac{C_n^\pm}{\zeta - \alpha_n}.$$

The function $f^\pm$ has simple poles at $\zeta = \beta_n$, $n \geq -2$ and at $\zeta = -\beta_0$, but no zeros at $\zeta = \alpha_n$, $n \geq 0$ contrary to the semi-infinite case. Note the polynomial of $\left(\gamma_1^\pm, \gamma_2^\pm\right)$ of $g^\pm(\zeta)$ determines the unique solution of the boundary value problem because of $f^\pm$ having a proper asymptotic behaviour. It can be shown that $f^\pm(\zeta) = O\left(\zeta^{-1}\right)$ as $|\zeta| \to \infty$ using the same argument in Appendix A of (Linton 2001), since $\alpha_n = \pi n/H + O\left(n^{-1}\right)$ and $\beta_n = \pi n/H + O\left(n^{-4}\right)$ as $|\zeta| \to \infty$.

Consider the contour integration and use the Cauchy's residue theory

$$\frac{1}{2\pi i}\int_{\Gamma}f^\pm(\zeta)\left(\frac{1}{\zeta - \alpha_m} \pm \frac{e^{-2\alpha_mL}}{\zeta + \alpha_m}\right)d\zeta = 0$$

where Γ is an integral contour that can be expanded to include all β_n's. The above equation is then equivalent to Eqn. 2.3, given that the following conditions are satisfied,

$$f^\pm(\alpha_m) \pm e^{-2\alpha_mL}f^\pm(-\alpha_m) = 0, \quad \text{for } m = 0, 1, \dots \tag{2.4}$$

Rewriting Eqn. 2.4 for $C_n^\pm$ gives

$$C_m^\pm \pm D_m\sum_{n=0}^{\infty}\frac{C_n^\pm}{\alpha_m + \alpha_n} = \pm D_m\left(\alpha_m^2 - \gamma_1^\pm\alpha_m + \gamma_2^\pm\right) \tag{2.5}$$

where

$$D_m = e^{-2\alpha_mL}\frac{2\alpha_m(\alpha_m + \beta_0)(\alpha_m - \beta_{-2})(\alpha_m - \beta_{-1})(\alpha_m - \beta_m)}{(\alpha_m - \beta_0)(\alpha_m + \beta_{-2})(\alpha_m + \beta_{-1})(\alpha_m + \beta_m)}$$
$$\times \prod_{n=0,n\neq m}^{\infty}\frac{(1 - \alpha_m/\beta_n)(1 + \alpha_m/\alpha_n)}{(1 + \alpha_m/\beta_n)(1 - \alpha_m/\alpha_n)}$$

for $m \geq 0$. Note that $C_m^\pm$ decays exponentially as m becomes large because D_m decays exponentially as m becomes large. Hence only a small number of $C_m^\pm$ will be required to obtain acceptable values for $f^\pm(\zeta)$. This is precisely the reason for the use of the residue calculus technique. The normalizing constant $G^\pm$ can be determined by the amplitude of the incident wave, which is set to be 1. It is possible to prove that the system of equations given by Eqn. (2.5) has a unique solution with $\sum_{m=0}^\infty (C_m^\pm)^2 < \infty$, using the method described in Appendix B of (Evans 1992).

From Eqn. 2.5, $\{C_n^\pm\}$ can be expressed as a polynomial of $\gamma_1^\pm$ and $\gamma_2^\pm$ with known coefficients. The coefficients $\{b_n^\pm\}$ can then be expressed as linear functions of $\gamma_1^\pm$ and $\gamma_2^\pm$. When the edges of the plate are free of shear forces and bending moment the solution satisfies

$$\left. \begin{array}{l} \Phi_{zxxx}^\pm - l^2\nu'\Phi_{zx}^\pm = 0, \\ \Phi_{zxx}^\pm - l^2\nu\Phi_z^\pm = 0, \end{array} \right\} \text{ at } x = -L - 0, z = 0 \quad (2.6)$$

where $\nu' = 2 - \nu$. Four linear equations of $\gamma_1^\pm$ and $\gamma_2^\pm$ can be formulated by substituting the modal expansion of $\Phi^\pm$ into Eqn. 2.6.

2.4 Coefficients of free surface

The coefficients α_n can be found from Eqn. 2.2 (instead of eliminating α_m), then

$$\sum_{n=0}^\infty V_n^\pm \left(\frac{1}{\beta_n + \alpha_m} \pm \frac{e^{-2\alpha_m L}}{\beta_n - \alpha_m} \right) = \frac{1}{\beta_n - \alpha_m} \pm \frac{e^{-2\alpha_m L}}{\beta_n + \alpha_m}$$
$$\mp \alpha_m a_m^\pm e^{-\alpha_m L} \sinh 2\alpha_m L.$$

The equivalent contour integration is

$$\frac{1}{2\pi i} \int_\Gamma f^\pm(\zeta) \left(\frac{1}{\zeta + \alpha_m} \pm \frac{e^{-2\alpha_m L}}{\zeta - \alpha_m} \right) d\zeta = 0.$$

This time, the resulting system of equations from the similar procedure as in the previous section is

$$\pm \frac{4\alpha_m a_m^\pm}{A_m B_0} e^{-\alpha_m L} \sinh 2\alpha_m L = f^\pm(-\alpha_m) \pm e^{-2\alpha_m L} f^\pm(\alpha_m).$$

Furthermore, because of Eqn. 2.4, the above equation becomes

$$a_m^\pm = \pm \frac{f^\pm(-\alpha_m) e^{-\alpha_m L} A_m B_0}{4\alpha_m}. \quad (2.7)$$

Note that the procedure shown here gives the formula for $a_m^\pm$ without repeating the mode matching shown in the previous section.

2.5 Semi-infinite plate

The semi-infinite plate can be simplified further, because there are no length correction terms ($\{C_n\}$). The complex function for the semi-infinite plate case is

$$f(\zeta) = \frac{G(\zeta^2 + \gamma_1\zeta + \gamma_2)}{(\zeta - \beta_0)(\zeta - \beta_{-2})(\zeta - \beta_{-1})} \prod_{n=1}^\infty \frac{1 - \zeta/\alpha_n}{1 - \zeta/\beta_n}.$$

The system of equations from the mode-matching are reproduced by the contour integrations,

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - \alpha_n} d\zeta = 0, \text{ for } \beta_n$$
$$\frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta + \alpha_n} d\zeta = 0, \text{ for } \alpha_n$$

The reflection coefficient is then given, with $\beta_{-2} = \sigma + i\tau$, $\alpha_0 = -i\alpha$ and $\beta_0 = -i\beta$, by

$$R = \frac{w_\gamma^-(\alpha)(\alpha - i\beta_0)}{w_\gamma^+(\alpha)(\alpha + i\beta_0)} \exp[2i\chi(\alpha)],$$
$$w_\gamma^\pm(\alpha) = \alpha^2 \pm i\gamma_1\alpha - \gamma_2,$$
$$\chi(\alpha) = \frac{\pi}{2} + \tan^{-1}\frac{\alpha + \tau}{\sigma} + \tan^{-1}\frac{\alpha - \tau}{\sigma}$$
$$+ \sum_{n=1}^\infty \left(\tan^{-1}\frac{\alpha}{\beta_n} - \tan^{-1}\frac{\alpha}{\alpha_n} \right).$$

The modulus of T can be obtained from the following simple relationship,

$$|R|^2 + \frac{\beta}{\alpha\omega^2}|T|^2 = 1.$$

The value of T is given by

$$T = \frac{2\alpha(\alpha - \beta)w_\gamma^+(\beta)}{A_0 B_0 w_\gamma^+(\alpha)} P,$$
$$P = \frac{(\alpha - i\beta_{-2})(\alpha - i\beta_{-1})}{(\beta - i\beta_{-2})(\beta - i\beta_{-1})} \prod_{n=1}^\infty \frac{(1 + i\beta/\alpha_n)(1 + i\alpha/\beta_n)}{(1 + i\beta/\beta_n)(1 + i\alpha/\alpha_n)}.$$

It is a straightforward procedure to derive the reflection and transmission coefficients for the case of plate to water, i.e., wave is incident from $x = \infty$. The complex function that is required for the solution is

$$f(\zeta) = \frac{G(\zeta^2 + \gamma_1\zeta + \gamma_2)}{(\zeta + \beta_0)(\zeta - \beta_{-2})(\zeta - \beta_{-1})} \prod_{n=0}^\infty \frac{1 - \zeta/\alpha_n}{1 - \zeta/\beta_n}.$$

The conservation of energy relationship now is

$$|R|^2 + \frac{\alpha_0\omega^2}{\beta_0}|T|^2 = 1.$$

Further simplification when $\delta = \theta = 0$ is given in (Linton and Chung 2002).

3 Results and analyses of the computation

The numerical computation requires minimal computation technique, which is one of the advantages of the RCT. Figs. 1 and 2 show the results of the computation with the reflection coefficients and the magnitude of the surface. The rapidly fluctuating feature 'spikes' seen in the high frequency region in Fig. 1 is due to 'resonance'. The curves of the reflection coefficient show a series of discrete frequencies that correspond to perfect transmission. Fig. 2 shows the magnitude of the surface deflection for $\omega = 0.2, 0.5, 1, 3$. The magnitude of the incident wave is one for all cases. Note that in the preceding sections the magnitude of the potential was one.

An advantage of the RCT over the direct mode-matching technique is that the coefficients $\{a_n^{\pm}\}$ can be determined using the same function $f^{\pm}(\zeta)$. The formulae expressed by the infinite products are stable and fast convergent due to the fast convergence of the roots of the dispersion equations. The formulae for the coefficients can easily be implemented to computer codes that are computationally efficient and stable for a wide range of physical parameters. The derivation of the formulae is analogous to that of the finite-dock problem. The finite-length correction terms are added to the solution for the semi-infinite problem. It is shown that the formula for R reverts back to the formula for the semi-infinite problem as $L/H \to \infty$. The two sets of the edge conditions are incorporated into the complex valued function as the polynomial of the second degree and its two unknown coefficients.
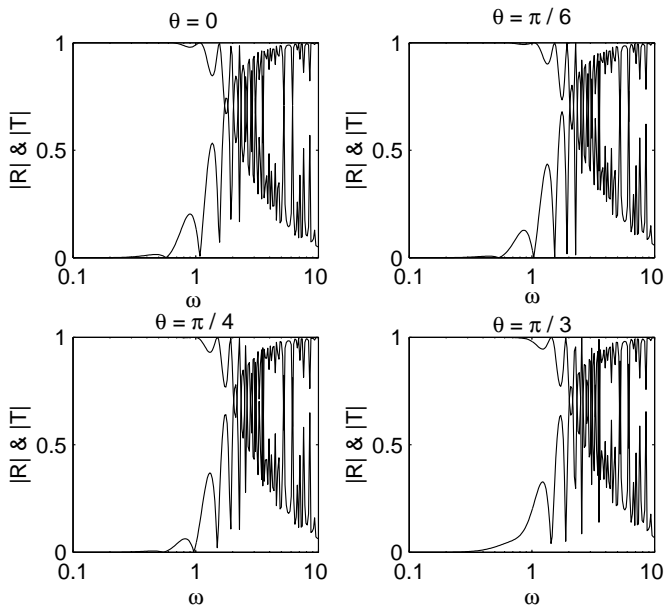

Figure 2: Magnitude of the surface deflection of the plates and the water for various frequency. The physical parameters are $H = L = \pi/3$, $\theta = 0$.

Figure 1: The reflection and transmission coefficients versus non-dimensional radial frequency ω (logarithmic scale). The incident angles are varied, $\theta = 0, \pi/6, \pi/4, \pi/6$. The parameters are $H = L = \pi/3$. Number of length correction terms is 3.

Incorporation of the second order polynomial into the complex function f assures the uniqueness of f (Liouville's theorem.) The order of the polynomial is determined by the asymptotic behaviour of the infinite products that involve the terms $(1 \pm \zeta/\alpha_n)$ and $(1 \pm \zeta/\beta_n)$, in other words, the positions of the roots of the dispersion equations. Therefore, one is able to determine the solution and the uniqueness of it only from the knowledge of the positions of the roots of the dispersion equations without resorting to the complicated theories of elliptic partial differential equations and functional analysis.

References

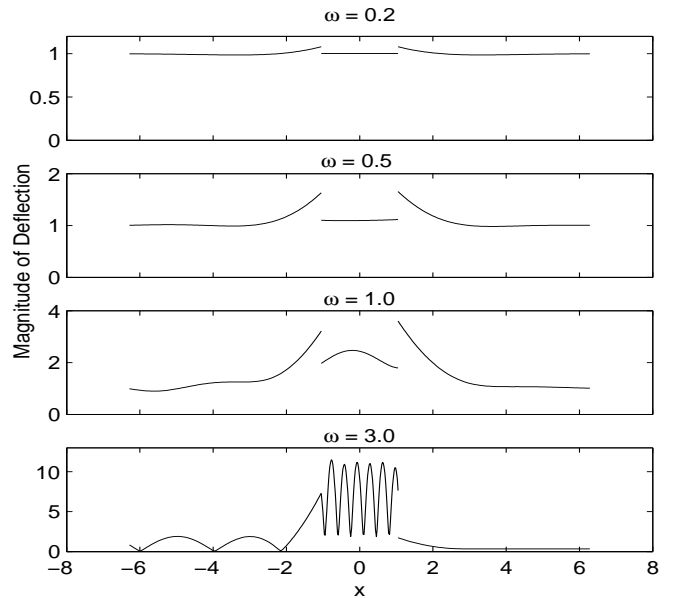Chung, H., and C. Fox. 2002. "Propagation of Flexural Waves at the Interface Between Floating Plates." *International Journal of Offshore and Polar Engineering* 12:163–170.

Evans, D.V. 1992. "Trapped Acoustic Modes." *IMA J. Appl. Math.* 49:45–60.

Evans, D.V., and T.V. Davies. 1968. *Wave-Ice Interaction Repot1313*. Hoboken: Davidson Laboratory, Stevens Institute of Technology.

Fox, C. 2000. "Scaling Laws for Flexural Waves in Floating Ice." *Proceedings of IUTAM Scaling Lows in Ice Mechanics and Ice Dynamics.*

Fox, C., and V.A. Squire. 1994. "On the Oblique Reflexion and Transmission of Ocean Waves at Shore Fast Sea Ice." *Phil. Trans. R. Soc. Lond* 347 (A): 185–218.

Lawrie, J.B., and I.D. Abrahams. 1999. "An Orthogonality Relation for a Class of Problems with High-Order Boundary Conditions; Applications in Sound-Structure Interaction." *Quarterly journal of applied mathematics* 52 (2): 161–181.

Linton, C.M. 2001. "The Finite Dock Problem." *ZAMP* 52:640–656.

Linton, C.M., and H. Chung. 2002. "Reflection And Transmission at the Ocean/Sea-Ice Boundary." *Wave motion (in press).*

Williams, T.D., and V.A. Squire. 2002. "Wave Propagation Across an Oblique Crack in an Ice Sheet." *International Journal of Offshore and Polar Engineering* 12:157–162.