DUAL EVOLUTION EQUATIONS FOR A WIND DRIVEN, BREAKING OCEAN WAVE AS DERIVED FROM VARIATIONAL CONSIDERATIONS

Marshall P. Tulin and Jiyue J. Li
Ocean Engineering Laboratory
University of California, Santa Barbara, CA 93106
mpt@engineering.ucsb.edu

ABSTRACT

Dual non-conservative evolution equations for a wind driven, breaking water wave are rigorously derived using the variational approach: a modified Hamiltonian principle involving the modulating wave Lagrangian plus a Work Function representing the non-conservative effects of external boundary stresses due to wind and breaking. The dual Euler equations represent conservation laws for <energy, momentum> or, correspondingly <energy, celerity>.

It is further shown that these dual equations correspond to the complex NLS, as modified by the non-conservative effects, thereby giving precise physical meaning to the NLS and providing its extension to the long time evolution of non-conservative ocean-like waves.

INTRODUCTION

Three quantities specify the state of a water wave in $(\vec{X}, T)$: the wave energy density, $E(\vec{X}, T)$; the wave momentum density, $M_i(\vec{X}, T)$; and the wave frequency functional, $\omega(k, E, \ldots)$; from these the wave phase speed, $C_i(\vec{X}, T)$, can be found.

Energetic ocean waves are known to be created and to grow due to energy and momentum pumping by the wind, $(\dot{e}_w; f_i)$, balanced by momentum and energy losses due to breaking, $(\dot{M}_{b_i}, D_b)$, accomplished by small overturning jets at the wave crests; the waves are known to be unstable to near neighbor sideband disturbances and to modulate, causing peaks in wave amplitude where breaking takes place; the dominant frequency is known to downshift steadily as the wave energy grows, and, correspondingly, the phase and group speeds increase. Nonlinear processes dominate.

A proper mathematical description of the appropriate conservation laws, including the external influences (wind pumping and wave breaking), explicating their influence on downshifting, has never been given. Here we give some new results of this nature, within the variational (Hamiltonian) framework initiated by Luke and Whitham (1974).

NEW RESULTS (General)

An appropriate description can be given in terms of the Whitham time averaged Lagrangian, $\mathcal{L}$, representing conservative mechanical effects, and a work function, W, introduced here, representing the non-conservative effects of external boundary stresses arising from the wind pressure and from the isolation of the overturning jet from the main wave. By application of the time-averaged Hamiltonian principle, eq. (9), the corresponding Euler equations are found to be:

$$\frac{\partial E}{\partial T}+\frac{\partial F_j}{\partial X_j}=\overline{\frac{\partial W}{\partial t}} \tag{1}$$

$$\frac{\partial M_i}{\partial T}+\frac{\partial S_{ij}}{\partial X_j}=-\overline{\frac{\partial W}{\partial x_i}} \tag{2}$$

and where it has been explicitly shown that:

$$\overline{\frac{\partial W}{\partial t}}(\vec{X},T)=\dot{e}_w-D_b \tag{3}$$

$$-\overline{\frac{\partial W}{\partial x_i}}(\vec{X},T)=f_i-\dot{M}_{b_i}=\dot{M}_i \tag{4}$$

where the variables X, T refer to the modulation scale (so-called long), the variables x_i, t to the wave oscillation (short), and the overbar to short time averaging.

In (1) and (2) we have identified and introduced the four physical dependent variables whose definitions follow from the Euler equations and are:

$$E(\text{energy})=\omega\mathcal{L},_{\omega}-\mathcal{L}+a_T\mathcal{L},_{a_T}\,;F_j(\text{energy flux})=\omega\mathcal{L},_{k_i}+a_T\mathcal{L},_{a_{X_i}} \tag{5}$$

$$M_i(\text{momentum})=-k_i\mathcal{L},_{\omega}-a_{X_i}\mathcal{L},_{a_T}\,;S_{ij}(\text{wave stress})=-k_i\mathcal{L},_{k_i}+\mathcal{L}\delta_{ij}-a_{X_i}\mathcal{L},_{a_{X_i}} \tag{6}$$

where $\mathcal{L}=\bar{L}$, and the Lagrangian L is,

$$L=\int_{-h_o}^{\eta}p\,dy=-\int_{-h_o}^{\eta}\left\{\Phi_t+\frac{1}{2}(\nabla\Phi)^2+gy\right\}dy \tag{7}$$

where it can be shown that the Hamiltonian principle for this non-conservative real ocean system is,

$$\int_R\int\left[\delta L+\delta W\right]d\vec{x}dt=0 \tag{8}$$

and the time averaged principle introduced by Luke and Whitham (1974) as modified here by the addition of the work function is:

$$\int_R\int[\delta\mathcal{L}+\delta\bar{W}]d\vec{x}dt=0 \tag{9}$$

where R is over the fluid domain excluding breaking jets, and W is determined from and satisfies the boundary conditions on the wave arising from external boundary stresses there, including those due both to wind pressure and due to the pressures and fluxes acting at the cut surfaces isolating the breaking jets from the main ocean wave.

From (5) and (6) it follows that,

$$C_i = \omega/k_i = \frac{E + \mathcal{L} - a_T \mathcal{L}_{,a_T}}{M_i + a_{X_i}\mathcal{L}_{,a_T}} = \frac{F_j - a_T \mathcal{L}_{,a_{X_i}}}{S_{ij} - \mathcal{L}\delta_{ij} + a_{X_i}\mathcal{L}_{,a_{X_i}}} \tag{10}$$

then utilizing (10) in (1) and (2), an evolution equation for C_i can be found, complementary to (1) and (2).

$$\begin{aligned} \frac{\partial C_i}{\partial T} &+ \left(\frac{S_{ij}}{M_i}\right)\frac{\partial C_i}{\partial X_j} = \frac{1}{M_i}\left\{\underline{\left[C_i \overline{\frac{\partial W}{\partial x_i}} + \overline{\frac{\partial W}{\partial t}}\right]}\right. \\ &\left. + \underline{\underline{\left[\frac{\partial}{\partial T}(\mathcal{L} - \mathcal{L}_{,a_T}\langle a_T + C_i a_{X_i}\rangle) + \frac{\partial}{\partial X_j}(C_i\delta_{ij}\mathcal{L} - \mathcal{L}_{,a_{X_j}}\langle a_T + C_i a_{X_i}\rangle)\right]}}\right\} \end{aligned} \tag{11}$$

The double underlined terms are quasi-periodic and provide for fluctuations of C_i within the wave group, and are essential for wave instability, modulation, and recurrence.

To the lowest relevant order in wave steepness, (ak), the contribution of wind pumping to the single underlined term disappears so that only the effect of wave breaking there is relevant; this can be parameterized in terms of the momentum loss and dissipation due to wave breaking and shown to be positive, Tulin (1996). Again to the lowest relevant order in (ak) the double underlined terms do not result in permanent downshifting, and therefore, long term frequency downshifting is controlled by wave breaking; the essential role of four wave energy transfer and the intercession of breaking in such transfer is discussed by Tulin and Waseda (1999).

NEW RESULTS (Specific)

The Lagrangian $\mathcal{L} = \mathcal{L}(\omega, k, a, \ldots)$ can be calculated from (7) by introducing a suitable approximation for (η, Φ). We consider one horizontal dimension only, $\vec{X} = X$, and take approximations for (η, Φ) suggested by multiple scale analyses:

$$\eta = a\cos\theta - a_T/\omega \cdot \sin\theta + 1/2ka^2\cos 2\theta + \ldots \tag{12}$$

$$\Phi = w/k(a\sin\theta - ya_X\cos\theta)e^{ky} + \ldots \tag{13}$$

Then L follows by substitution in (7), and $\mathcal{L}$ by subsequent integration over one cycle (2π) in the phase, $\theta = kx - \omega t$. As calculated by J.J. Li:

$$\begin{aligned} \mathcal{L} = 1/4ga^2(\omega^2/gk - 1) \quad &- \quad \frac{1}{8}gk^2a^4\frac{1+\omega^2/gk}{2} - \frac{1}{4}\frac{g}{\omega^2}(a_T)^2[2\omega^2/gk - 1] \\ &- \quad \frac{1}{8}\frac{\omega^2}{k^3}(a_X)^2 - \frac{1}{8}\frac{\omega^2}{k^3}aa_{XX}. \end{aligned} \tag{14}$$

The evolution equations are then found to be, in terms of e and C_g, the group velocity:

$$\frac{\partial e}{\partial T} + \frac{\partial(C_g e)}{\partial X} = \dot{e}_w - D_b \qquad , \qquad E \equiv e = \frac{1}{2}ga^2 \tag{15}$$

$$\frac{\partial C_g}{\partial T} + C_g \frac{\partial C_g}{\partial X} = \underline{\underline{\frac{1}{4} k \frac{\partial e}{\partial X} + \frac{C_g^2}{8k^2} \frac{\partial}{\partial X}\left(\frac{a_{XX}}{a}\right)}} + \underline{\frac{C_g}{e} \cdot \gamma D_b} \tag{16}$$

The underlined terms in (11) and (16) correspond; $\gamma D_b = C\dot{M}_b - D_b = 0(1) > 0$.

Intermediate results are the evolution equation for wave momentum,

$$\frac{\partial (e/C)}{\partial T} + \frac{\partial (e/2)}{\partial X} = -\frac{\overline{\partial W}}{\partial x} = f_x - \dot{M}_b \tag{17}$$

and the definition,

$$C_g = C/2[1 - \alpha_X/2k] = C/2[1 - (ak)^2/4 - a_{XX}/8ak^2] \tag{18}$$

where $\alpha(X,T)$, a fluctuating phase, is subsequently defined below.

The energy equation, (15), is correct through $0(ak)^4$, and the group velocity evolution, (16), through $0(ak)^3$, where (ak) is the wave steepness, a measure of nonlinearity.

These evolution equations, (15) and (16), can be combined and constructed in the complex form:

$$A_T + C_g A_X + i\frac{C_g}{4k} A_{XX} + \frac{i}{2}\omega k^2 /A/^2 A = A\left[\frac{(\dot{e}_w - D_b)}{g/A/^2} - i4\gamma \int \frac{k D_b dX}{g/A/^2}\right] \tag{19}$$

The LHS coincides with the nonlinear Schroedinger equation, except that this relation is valid for all (X,T), i.e. it is not restricted to a neighborhood (k_o, C_{g_o}); in (19), $A = ae^{i(\theta+\alpha)}$ where $\alpha(\bar{X},T)$ is a fluctuating phase ($\theta = kx - \omega t$, as before). The RHS of (19) provides the real effects, including the downshifting which arises from the term involving the integral in X. For parameterization of $\dot{e}_w, D_b$ in terms of $/A/^2$ see Tulin, (1996), where a heuristic version of (1), (2), and (11) is also given in the case of planar waves; in this description, $\dot{e}_w \sim g/A^2/, D_b \sim g/A/^4$, where the constants of proportionality follow from field observations. The proper extension of the NLS to real waves and long fetches is therefore accomplished, and its actual physical nature, in the form of the two conservation laws, (15) and (16), is revealed.

References

[1] Tulin, M.P.(1996), "Breaking of Ocean Waves and Downshifting," in *Waves and Nonlinear Processes in Hydrodynamics*, pp 177–190, ed. Grue, et al., Kluwer Academic Publishers.

[2] Tulin, M.P. and Waseda, T. (1999), "Laboratory observations of wave group evolution, including breaking effects," *JFM*, vol. 378, pp. 197–232.

[3] Whitham, G.B. (1974), *Linear and Nonlinear Waves*, Wiley.

Discussion Sheet

Abstract Title :	Dual evaluation equations for a wind driven, breaking ocean wave as derived from variational considerations.		
(Or) Proceedings Paper No. :	47	***Page :***	183
First Author :	Tulin, M.P.		
Discusser :	John Grue		

Questions / Comments :

I have a comment and a question:
The comment is related to the NLS-equation. Fully nonlinear simulations of large wave events exhibit more frequent occurance of the large waves, than simulations using the NLS-equation, see paper number 8 in this workshop (Clamond, D. & Grue, J.). Thus, the fully nonlinear simulations exhibit fundamental differences from the NLS-eq. This supports the use of your eq. (19).

My question is : Could you comment on extensions to 3-D,
a) with regard to your equations, and
b) with regards to physical interpretation?

Author's Reply :
(If Available)

Thank you for your question. One of the reasons we have carried out the development is to provide a basis for the derevation of evolution equations in 3-D and of higher order, but we have ourselves not done either. It is a good idea to do so, however.