Radiation problem for an interface-piercing cylinder in a two-layer fluid

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SUMMARY

The linear 2-D water-wave problem, describing small oscillations of a horizontal cylinder is considered. The cylinder is submerged in a stratified unbounded fluid, which consists of two layers of different density and infinite depth, and intersects the interface. A system of boundary integral equations for the problem is derived, behaviour of its solution near the intersection points is studied. Asymptotic formulae for the added mass and damping coefficients at low frequency are derived. In the case of a circular cylinder the water-wave problem is solved by the method of multipole expansion. The added mass and damping coefficients are calculated and essential dependence of these values on a frequency of oscillations, on the difference of densities and on the position of the cylinder with respect to the interface is shown. By the relation between the scattering and radiation potentials by McIver (1996), the reflection coefficient is defined for the corresponding diffraction problem.

1. STATEMENT OF THE PROBLEM

We use the Cartesian coordinates (x, y), where the vertical coordinate y is measured upwards. The interface $y = 0$ separates the upper fluid with density ρ_1 occupying the domain $L^{(1)}$ ($|x| < \infty, y > 0$) and the fluid with density $\rho_2 = \rho_1(1+\varepsilon)$ ($\varepsilon > 0$) filling in the lower layer $L^{(2)}$ ($|x| < \infty, y < 0$). The fluids are assumed to be inviscid and incompressible. The flow in each layer is potential and the interface-piercing cylinder undergoes small oscillations in three possible degrees of freedom (sway, heave and roll) at a frequency ω. Body's cross-section is denoted by B and its wetted contour $\Gamma = \partial B$ is divided by the interface into two parts $\Gamma = \Gamma^{(1)} \cup \Gamma^{(2)}$ and intersects the interface at two points $P_\pm = (\pm b, 0)$. The one-side tangent to $\Gamma^{(s)}$ at $P_\pm$ and the interface form the angle $\beta_\pm^{(s)}$ measured through the fluid (here and subsequently the superscript s is equal to 1 for the upper layer and 2 for the lower one). The two-layer fluid represents a limiting case of a steep pycnocline. The radiation and diffraction problems for a circular cylinder located beneath a pycnocline in a constant-density layer have been considered by Sturova (1999).

The disturbed oscillatory motion of the fluid is assumed to be steady, and the velocity potentials can be written as

$$\Phi^{(s)}(x, y, t) = \mathrm{Re}\Big[\mathrm{i}\omega \sum_{j=1}^{3} \eta_j \phi_j^{(s)}(x, y) \exp(\mathrm{i}\omega t)\Big],$$

where $\phi_j^{(s)}$, $j = 1, 2, 3$, are radiation potentials due to motions of the cylinder with unit amplitude in the three degrees of freedom; η_j are corresponding motion amplitudes.

The radiation potentials satisfy the Laplace equation inside the domains occupied by the fluid

$$\Delta\phi_j^{(s)} = 0, \quad (x, y) \in W^{(s)} = L^{(s)} \setminus B, \quad s = 1, 2. \tag{1}$$

The linearized kinematic and dynamic boundary conditions on the interface $(y = 0)$ outside the body are

$$\frac{\partial\phi_j^{(1)}}{\partial y} = \frac{\partial\phi_j^{(2)}}{\partial y}, \quad (1+\varepsilon)\nu\phi_j^{(2)} - \nu\phi_j^{(1)} = \varepsilon\frac{\partial\phi_j^{(1)}}{\partial y}, \quad \nu = \frac{\omega^2}{g}, \tag{2}$$

respectively, where g is the acceleration due to gravity. Besides, far from the interface we have

$$\nabla \phi_j^{(s)} \to 0 \ \text{ as } \ (-1)^{s+1} y \to +\infty, \quad s = 1, 2. \tag{3}$$

The boundary condition on body's wetted contour is as follows:

$$\frac{\partial \phi_j^{(s)}}{\partial n} = n_j, \quad (x, y) \in \Gamma^{(s)}, \tag{4}$$

where $\boldsymbol{n} = (n_x, n_y)$ is the inward normal to the contour Γ. We shall use the notation $n_1 = n_x$, $n_2 = n_y$, $n_3 = (y - y_0) n_1 - (x - x_0) n_2$, where x_0, y_0 are coordinates of the centre of the roll oscillations. In the far field a radiation condition should be imposed that requires the radiated waves to be outgoing. Besides, since the domains $W^{(s)}$ have corner points, where solution to (1)–(4) can have strong singularities, local finiteness of energy should be preposed

$$\int_{E \cap W^{(s)}} \left| \nabla \phi_j^{(s)} \right|^2 \mathrm{d}x \, \mathrm{d}y < \infty. \tag{5}$$

Here E is a compact domain including $P_\pm$.

The hydrodynamic loads by the oscillations $\boldsymbol{F} = (F_1, F_2, F_3)$ are typically written in matrix form, and, omitting the hydrostatic term, we have

$$F_k = \sum_{j=1}^{3} \eta_j \tau_{kj}, \quad \tau_{kj} = \omega^2 \sum_{s=1}^{2} \rho_s \int_{\Gamma^{(s)}} \phi_j^{(s)} n_k \, \mathrm{d}\Gamma = \omega^2 \mu_{kj} - \mathrm{i} \omega \lambda_{kj}, \tag{6}$$

where μ_{kj} and λ_{kj} are the added mass and damping coefficients, respectively.

2. BOUNDARY INTEGRAL EQUATIONS

Solution to (1)–(5) for the body of arbitrary shape is sought in the form of a single layer potential. We write

$$\phi_j^{(s)}(z) = \sum_{\ell=1}^{2} \int_{\Gamma^{(\ell)}} \sigma_j^{(\ell)} G^{(s,\ell)}(z, \zeta) \, \mathrm{d}\Gamma_\zeta, \quad z = x + \mathrm{i}y, \quad \zeta = \xi + \mathrm{i}\eta, \quad s = 1, 2,$$

where $\sigma_j^{(\ell)}$ is an unknown density and $G^{(s,\ell)}(x, y; \xi, \eta)$ is the Green function of the problem (1)–(3), s and ℓ are such that $(x, y) \in L^{(s)}$, $(\xi, \eta) \in L^{(\ell)}$. Expressions for the Green function are given by Gorgui & Kassem (1978).

From condition (4) we arrive at the system of integral equations

$$\pi \, \sigma_j^{(s)}(z) - \sum_{\ell=1}^{2} \int_{\Gamma^{(\ell)}} \sigma_j^{(\ell)}(\zeta) \frac{\partial G^{(s,\ell)}}{\partial n_z}(z, \zeta) \, \mathrm{d}\Gamma_\zeta = n_j(z), \quad s = 1, 2. \tag{7}$$

The operator of the system is not compact in L_2, but it is Fredholm's one in the weight Banach space of continuous in $\mathrm{int}\Gamma^{(s)}$ functions having the finite norm $||\sigma||_\kappa = \sup\left\{|y|^{1-\kappa}|\sigma(z)| : z \in \mathrm{int}\Gamma^{(s)}\right\}$, where $0 \leqslant \kappa \leqslant 1$ such that $\max\limits_{s=1,2} \max\limits_{\pm}\left\{\sin \kappa |\pi - 2\beta_\pm^{(s)}| / \sin \kappa\pi\right\} < 1$.

Using general theorems on asymptotics of solutions in domains with non-smooth boundary (see e.g., Nazarov & Plamenevsky, 1994) we find asymptotics of $\phi_j^{(s)}$ and the densities $\sigma_j^{(s)}(z)$ at the end points of $\Gamma^{(s)}$. We consider the left point $P = P_- = (-b, 0)$, omitting the subindex, so that e.g. $\beta^{(s)} = \beta_-^{(s)}$. Let (r, ϑ) be the polar coordinates with centre at P, such that $x = -b - r\cos\vartheta$, $y = -r \sin\vartheta$. Then,

$$\sigma_j^{(s)}(r) \sim a^{(s)} + b^{(s)} r^{\lambda^{(s)}}, \quad \lambda^{(s)} = \frac{2\beta^{(s)} - \pi}{2(\pi - \beta^{(s)})},$$

where $a^{(s)}$, $b^{(s)}$ are some (non-zero, in general case) constants. From the latter formula it follows that for $\beta_\pm^{(s)} < \frac{\pi}{2}$ the function $\sigma_j^{(s)}$ generally has a weak singularity at P with exponent $\lambda^{(s)} \in (-1/2, 0)$. This fact should be taken into account, in particular, when the system (7) is solved numerically.

3. MULTIPOLE EXPANSION METHOD

In order to solve the problem for the case of the circular cylinder the method of multipole expansions is used by analogy with the paper by Eatock Taylor & Hu (1991), where the 2-D and 3-D wave diffraction and radiation problems were considered for the bodies floating on a free surface of a homogeneous fluid.

Let the circular cylinder contour Γ be defined as $x^2 + (y-h)^2 = a^2$, where a is its radius and h is the vertical coordinate of cylinder's centre. For the circular cylinder only horizontal $(j = 1)$ and vertical oscillations $(j = 2)$ should be considered. We write a solution to (1)–(5) in the form

$$\phi_1^{(s)} = \sum_{m=1}^{\infty} A_m S_m^{(s)}, \quad \phi_2^{(s)} = \sum_{m=0}^{\infty} B_m C_m^{(s)}, \tag{8}$$

where $S_m^{(s)}$, $C_m^{(s)}$ are anti-symmetric and symmetric multipole potentials, respectively, satisfying all the conditions of the problem except (4); the latter is used to find the unknown coefficients A_m and B_m.

We introduce two auxiliary polar systems of coordinates (r, θ) and (R, τ):

$$r = \sqrt{x^2 + (y-h)^2}, \quad \theta = \arctan[x/(y-h)], \quad R = \sqrt{x^2 + (y+h)^2}, \quad \tau = \arctan[x/(y+h)].$$

Then, the multipole potentials can be written as follows

$$S_1^{(i)} = -a e_{j+1} k_0 V_s^{(i)}, \quad S_1^{(j)} = a\left(\frac{\sin\theta}{r} + \frac{\sin\tau}{R} + e_{j+1} k_0 V_s^{(j)}\right), \quad S_m^{(i)} = \pm \frac{e_{j+1} k_0 a^m}{m-1} \frac{\sin(m-1)\theta}{r^{m-1}},$$

$$S_m^{(j)} = a^m \left\{ \frac{\sin m\theta}{r^m} - (-1)^m \frac{\sin m\tau}{R^m} + \frac{k_0}{m-1}\left[\pm \frac{\sin(m-1)\theta}{r^{m-1}} + (-1)^m e_1 \frac{\sin(m-1)\tau}{R^{m-1}}\right]\right\},$$

$$C_0^{(i)} = e_{j+1}\left[\ln\frac{r}{a} + V_c^{(i)}\right], \quad C_0^{(j)} = \ln\frac{r}{a} \pm e_1 \ln\frac{R}{a} - e_{j+1} V_c^{(j)}, \quad C_1^{(i)} = \mp a e_{j+1} k_0 \ln\frac{r}{a},$$

$$C_1^{(j)} = a\left[\frac{\cos\theta}{r} - \frac{\cos\tau}{R} + k_0\left(\mp \ln\frac{r}{a} - e_1 \ln\frac{R}{a}\right)\right], \quad C_m^{(i)} = \pm \frac{e_{j+1} k_0 a^m}{m-1} \frac{\cos(m-1)\theta}{r^{m-1}},$$

$$C_m^{(j)} = a^m \left\{ \frac{\cos m\theta}{r^m} + (-1)^m \frac{\cos m\tau}{R^m} + \frac{k_0}{m-1}\left[\pm \frac{\cos(m-1)\theta}{r^{m-1}} - e_1 (-1)^m \frac{\cos(m-1)\tau}{R^{m-1}}\right]\right\}.$$

Here $m \geqslant 2$, $k_0 = \nu/e_1$, $e_1 = \varepsilon/(2+\varepsilon)$, $e_2 = 2/(2+\varepsilon)$, $e_3 = (1+\varepsilon)e_2$ and we use the indices $i = 3/2 \mp 1/2$, $j = 3/2 \pm 1/2$. Under this notation in the preceding formulas we should fix either lower or upper sign in symbols $\pm, \mp$. The upper (lower) signs correspond to the case when the centre of the cylinder is located in the upper (lower) layer. Also,

$$V_s^{(p)}(x,y) = p\,v \int_0^\infty \frac{\mathrm{e}^{kY_p}}{k-k_0} \sin kx \,\mathrm{d}k - \mathrm{i}\pi \mathrm{e}^{k_0 Y_p} \sin k_0 x$$
$$= \mathrm{Im}\big[\mathrm{e}^{Z_p} E_1(Z_p)\big] + \pi \mathrm{e}^{k_0 Y_p} \operatorname{sign}(x)\big(\cos k_0 x - \mathrm{i} \sin k_0 |x|\big),$$

$$V_c^{(p)}(x,y) = p\,v \int_0^\infty \frac{\mathrm{e}^{kY_p}}{k-k_0} \cos kx \,\mathrm{d}k - \mathrm{i}\pi \mathrm{e}^{k_0 Y_p} \cos k_0 x = \mathrm{Re}\big[\mathrm{e}^{Z_p} E_1(Z_p)\big] - \mathrm{i}\pi \mathrm{e}^{k_0 Y_p}\big(\cos k_0 x - \mathrm{i} \sin k_0 |x|\big),$$

where $p\,v$ indicates the principal-value integration, $Y_p = (-1)^p y - |h|$, $Z_p = k_0\big[\mathrm{i}x + Y_p\big]$, $p = 1, 2$, and E_1 is the exponential integral.

Substituting the series (8) into the boundary condition (4), sequentially multiplying it by $\sin n\theta$ and $\cos n\theta$ $(n \geqslant 0)$ and integrating over the cylinder surface, we obtain an infinite system of linear equations for finding unknown coefficients A_m and B_m. In the present work this system is solved numerically by the reduction method. Finding A_m and B_m allows us to compute other characteristics of fluid motion for the radiation and diffraction problems. The damping coefficients λ_{jj} can be found either from (6) or by the energy relation in terms of far-field waves amplitudes (see Eatock Taylor & Hu, 1991). The reflection coefficient R_d in the diffraction problem is defined by the formula $R_d = \frac{1}{2}\big[B_0/B_0^* - A_1/A_1^*\big]$, where $*$ means complex conjugate.

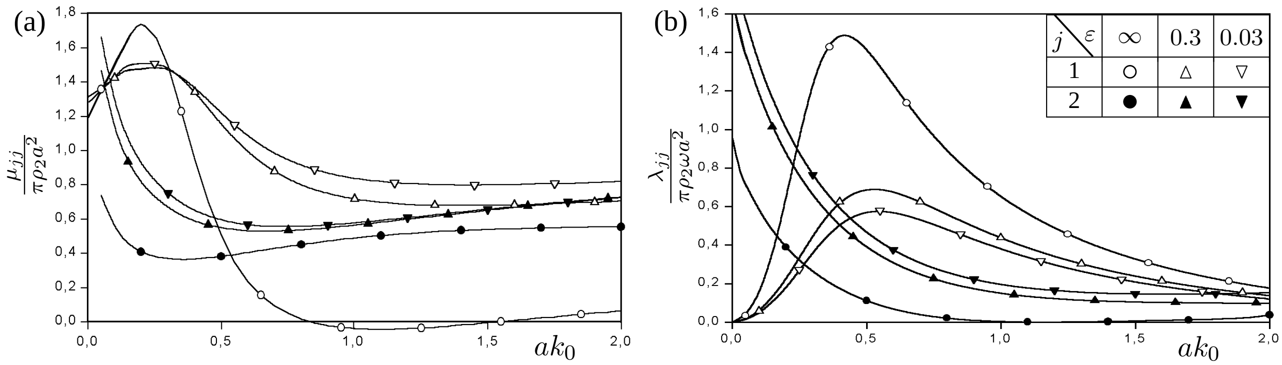

Figure 1: Added mass (a) and damping (b) coefficients for $h/a = -0.5$.

4. NUMERICAL RESULTS

As a test for the suggested numerical scheme we use the results of Greenhow & Ahn (1988) and Eatock Taylor & Hu (1991), who computed hydrodynamical loads for the circular cylinder partially submerged in the homogeneous fluid having the free surface. The same configuration with the cylinder partially submerged in the lower layer appears in the limit case $\varepsilon \to \infty$ of the considered problem with the two-layer fluid. For $h/a = -0.5$ the maximum difference between our computations, where 50 members of the series (8) are taken, and the values of hydrodynamical loads given as per the table by Eatock Taylor & Hu (1991) does not exceed 1%. At low frequency the numerical results are also in a good agreement with the asymptotic formulae (not presented here for economy of space) which we obtained for the added mass and damping coefficients.

Some of the numerical results are presented in Fig. 1, where hydrodynamic loads for the homogeneous $(\varepsilon = \infty)$ and two-layer fluids $(\varepsilon = 0.3; 0.03)$ are compared. It is observed that for oscillations of interface-piercing cylinder in the two-layer fluids, the values of added mass coefficients as a rule exceed the values computed for the same cylinder piercing the free surface of homogeneous fluid filling in the lower layer. It is found that for the homogeneous fluid and $-1.1 \leqslant h/a \leqslant -0.5$ the added mass coefficient μ_{11} can have negative values, but this feature was not observed for either of the considered two-layer fluids.

5. CONCLUSION

For a body of arbitrary shape a system of boundary integral equations is derived which can be used to find a solution to the problem in the form of a single layer potential. It is shown that the potential density can have a singularity of defined type at the contour points belonging to the interface. For the circular cylinder the problem is solved by the multipole expansion method. Influence of stratification and position of the cylinder at the interface is shown to be essential for radiation loads and reflection coefficient in the diffraction problem. Unlike the case of the cylinder totally submerged in one of the layers the radiation loads depend on the type of oscillations, and the reflection coefficient is not identically zero. The results can be extended to the 3-D case.

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