IMPULSIVE 3-D GREEN FUNCTION FOR A SLOPING BEACH

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SUMMARY. The impulsive flow due to a concentrated flux through a beach with a constant slope is investigated. A general integral solution for the Green function is obtained. Closed form, asymptotic and numerical results are presented.

1 Introduction

Due to the current growing interest among geophysicists, hydrodynamicists and coastal engineers in landslides and submarine slumping-induced tsunami (e.g., Tinti et al. 2001), a simplified model is proposed to study the initial free-surface deflection caused by an impulsive underwater mass failure on a sloping beach.

The potential hazards of tectonic bottom motions on near-shore floating and fixed structures have been discussed by Miloh & Striem (1978) within the framework of non-linear solitary wave theory. Among the early attempts to solve the linear tsunami generation problem, we also mention Tuck & Hwang (1972) and Hammack (1973).

It is commmon to assume that the earthquake triggered bottom motion is impulsive, and on the free-surface the Dirichlet equi-potential type of boundary condition can be applied. Under such an assumtion one can relate the shape, size and impulsive velocity of the moving underwater slumping to an equivalent source distribution over the undisturbed slope. For such a purpose thin/flat/slender body approximations can be invoked combined with the Harbitz (1992) model.

In this respect the so-called impulsive Green function directly determines the free-surface deflection in terms of the bottom forcing and beach slope. To determine this Green function it is reasonable to use the small-time expansion (Tyvand & Storhaug 2000). For the case of a sloping beach with a constant slope, one can go a step further and derive an exact expression for the impulsive Green function by employing the Kontorovich-Lebedev integral transform (Miloh et al, 2002). Once such a problem is solved, the landslide-induced non-linear problem can be then represented as an asymptotic sequence of linearized problems using time as a small parameter. The special properties of the impulsive Green functions are presented in the sequel.

2 Boundary-value problem

We consider the free-surface flow generated by a point on an impermeable rigid boundary. The fluid is inviscid, incompressible and its motion is irrotational. A Cartesian coordinate system $Oxyz$ is introduced, with the Oxy plane taken in the undisturbed free surface, whereas the z axis points vertically upwards.

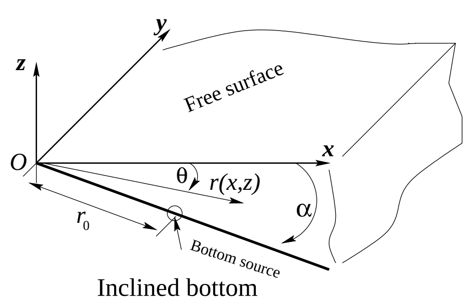

Figure 1: System of coordinates and general notations.

The impulsive Green function $G(x, y, z; x_0, y_{0,} z_0)$ is a harmonic function which is defined by the boundary conditions

$$G(x, y, 0; x_0, y_0, z_0) = 0, \tag{1}$$

$$\frac{\partial G}{\partial n} = 2\pi\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \tag{2}$$

and a proper decay at infinity. Here $\mathbf{n}$ is the normal to the rigid boundary at the location of the source and directed into the fluid. This Green function can be represented as a sum of singular and regular parts as $G = -1/R + H$, where $R = \sqrt{(x-x_0) + (y-y_0)^2 + (z-z_0)^2}$ and $H(x, y, z; x_0, y_0, z_0)$ is a regular harmonic function satisfying the boundary conditions (1) and (2). The free surface vertical velocity W and the impulsive Green are related by:

$$W(x, y; x_0, y_0, z_0) = \left.\frac{\partial}{\partial z} G(x, y, z; x_0, y_0, z_0)\right|_{z=0}. \tag{3}$$

3 Green function

For some particular geometries the Green function can be constructed by using certain images of the basic singularity in such a way that the required boundary condition on the free surface and rigid boundaries are satisfied. For instance, for a beach with slope angle $\alpha = \pi/2l$, (l–integer) the free surface velocity is represented as

$$W(x, y; r_0\cos\alpha, 0, -r_0\sin\alpha) = \sum_{n=0}^{2l-1} \frac{(-1)^n r_0 \sin(2n+1)\alpha}{\left[x^2 + r_0^2 + y^2 - 2xr_0\cos(2n+1)\alpha\right]^{3/2}}. \tag{4}$$

Consider a uniform beach with an arbitrary constant slope α. By introducing cylindrical coordinates $x = r\cos\theta, z = -r\sin\theta$, and using the condition of zero normal velocity at the slope allows us to construct the general solution in the form of the Kontorovich-Lebedev integral (Lebedev et al, 1965):

$$H(r, y, \theta; r_0, 0, \alpha) = \frac{2}{\pi}\int_0^\infty \int_0^\infty \frac{\cosh[(\alpha-\theta)\tau]}{\cosh(\alpha\tau)} \cos ky \; K_{i\tau}(kr) \; dk \; d\tau, \tag{5}$$

where K_ν denotes the modified Bessel function of the second kind, and $A(k,\tau)$ can to be determined from the equipotential boundary condition as:

$$A(k,\tau) = \frac{2}{\pi}\cosh[(\pi-\alpha)\tau] K_{i\tau}(kr_0), \tag{6}$$

Finally, we end up with the following expression for the surface velocity Green function W:

$$\begin{aligned} W &= -\frac{\partial}{\partial z}\left(\frac{1}{R}\right)\Bigg|_{z=0} - \frac{i}{\pi x\sqrt{2xr_0}}\int_\lambda^\infty \frac{ds}{\sqrt{\cosh s - \cosh\lambda}} \\ &\quad \times \int_{-\infty}^{\infty} \frac{\tau\sinh\alpha\tau\cosh[(\pi-\alpha)\tau]}{\sinh\pi\tau\cosh\alpha\tau} e^{is\tau}\, d\tau, \end{aligned} \tag{7}$$

where $\cosh\lambda = (x^2 + y^2 + r_0^2)/2xr_0$.

4 Closed form expressions

Here the inner integral can be calculated explicitly by choosing the contour of integration in the upper complex half-plane $\psi = \tau + i\tau_1$ as it is shown in Fig. 2a. To calculate the residues we note that in the complex plane ψ the integrand has single poles at the roots of the equations $\sinh\pi\tau = 0$, $\tau_k = ik$, $k = (1, 2\cdots)$, $\cosh\alpha\tau = 0$, $\tau_n = \frac{2n+1}{2\alpha}\pi i$, $(n = 0, 1, \cdots)$. Calculating the residues leads to series which can be calculated in a closed form. Substituting into (7) and performing one analytic integration yields the following expression for the impulsive free-surface velocity Green function for a uniform sloping beach:

$$W = \frac{\pi}{2x\sqrt{2xr_0}\alpha^2}\int_\lambda^\infty \frac{\sinh(\pi s/2\alpha)}{\cosh^2(\pi s/2\alpha)} \frac{ds}{\sqrt{\cosh s - \cosh\lambda}}. \tag{8}$$

This result is continuous with respect to α and is valid for any $0 < \alpha \le \pi$.

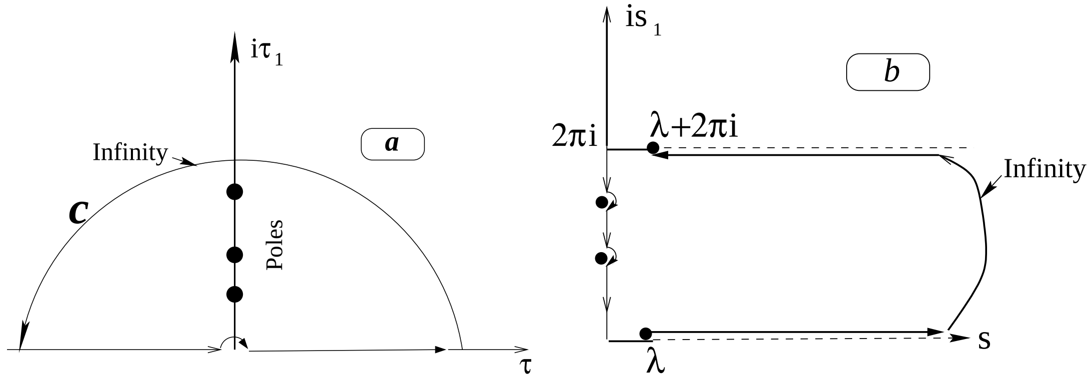

Figure 2: Contours of integration.

For some particular cases, i.e., $\alpha = \pi, \pi/2$ and $\pi/4$, the integral (8) can be calculated straightforwardly. To calculate it for $\alpha = \pi/2l$ (l-integer) we perform the integration in the complex plane $s + is_1$ along the contour C shown in Fig. 2b. The integrand has second-order poles on the imaginary axis at points which are defined by the roots of the equation $\cosh\frac{\pi s}{2\alpha} = 0,\ s_m = \frac{2m+1}{2l}\pi i,\ (m = 0, 1, \cdots)$. Since the imaginary poles are located within the range $\mathrm{Im}\, s_m \le 2\pi$, it follows that the number of poles inside the contour is finite and less than $2l - 1$. On the imaginary axis (excluding the poles) and on the intervals $[0, \lambda]$ and $[2\pi i, \lambda + 2\pi i]$ the integral is imaginary. On the lower cut $(\lambda + i0, \infty + i0)$ and the upper cut $[\infty + 2\pi(i - 0)], \lambda + 2\pi(i - 0)$ the integrals are equal. Thus, calculating the residues leads to series (4) obtained by the method of images.

5 Asymptotics

For $\lambda/\alpha \gg 1$ it follows that $\exp(-\frac{\pi\lambda}{2\alpha}) \ll 1$. In such a case the expression for the free surface Green function (8) can be calculated in a closed form:

$$W = \frac{\pi}{x\sqrt{xr_0}\alpha^2} Q_\nu(\cosh\lambda), \tag{9}$$

where $\nu = \pi/2\alpha - 1/2$, and Q_ν is the Legendre function of the second kind. This expression can be used to calculate the free-surface velocity Green function for any angles $\alpha \ll \lambda$. Invoking the asymptotic expressions of $Q_\nu(\cosh\lambda)$ for small α and large λ, it is illustrated that for small angles α the free-surface velocity Green function exhibits a boundary layer behavior with an exponential decay $W \sim \alpha^{-3/2}\exp(-\pi\lambda/2\alpha)$. Near the shore line ($x \ll 1$) and for any $\alpha < \pi/2$, the free-surface velocity W decays to zero as $(x/r_0)^{\pi/2\alpha - 1}$; for $\alpha = \pi/2$, W is finite, and for $\pi/2 < \alpha \le \pi$ it is singular.

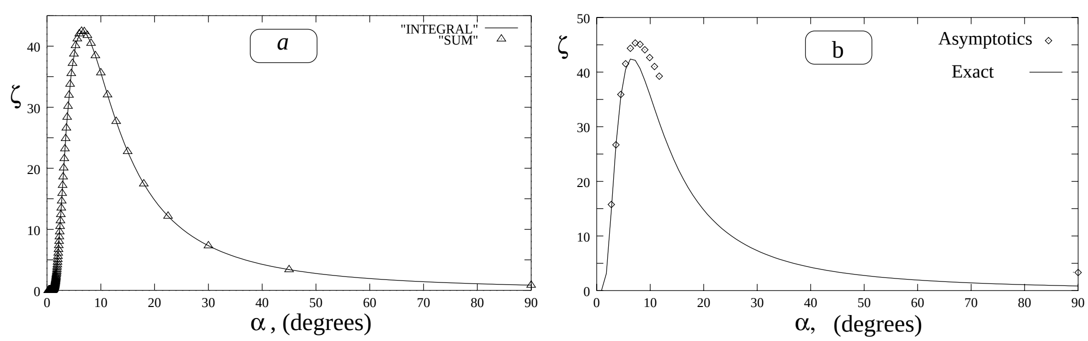

Figure 3: Free-surface velocity $\zeta = Wr_0^2/Q_0$ for a source of strength Q_0 located at $\xi = x/r_0 = 0.88, \eta = y/r_0 = 0$. Solid line: numerical integration of the integral representation (8) which is valid for any α. Symbols: a)–finite sum (4) which is valid for $\alpha = \pi/2n$, $n-$ integer; b)–asymptotic expression.

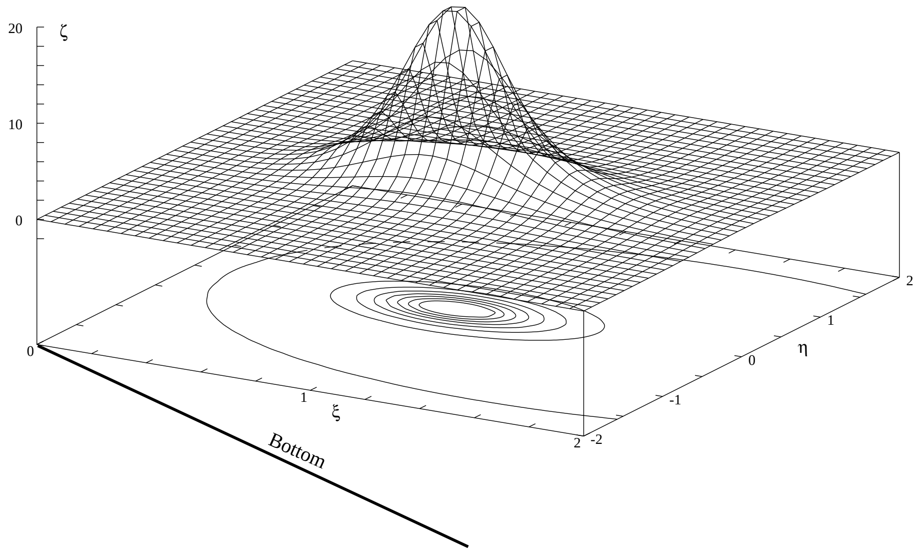

Figure 4: Free-surface velocity $\zeta = Wr_0^2/Q_0$ for a source of strength Q_0 as a function of $\xi = x/r_0$ and $\eta = y/r_0$: ($\alpha = \pi/10 = 18^o$.)

6 Numerical results

Fig. 3a is a comparison between the computed results of the surface velocity by invoking (8) and (4) which appears to be in a good agreement. In Fig. 3b the same data are compared against the asymptotic solution (9) where it is shown that for small sloping angles the asymptotic expression is quite accurate. In Fig. 4, a 3D plot of the free-surface velocity is shown. It is seen that the free surface velocity decays exponentially with respect to small α and small x. Its maximum appears approximately above the source.

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