Spectral Theory for a Floating Maseless Thin Plate on Water of Arbitrary Depth.

Michael H. Meylan and Christophe Hazard

Massey University, New Zealand and ENSTA, France.

1 Introduction

The floating thin plate is amongst the best studied problems in hydroelasticity. The time-harmonic linear wave response of a floating thin plate can be determined straightforwardly by a number of different methods.. Since the problem is linear, the time-dependent response of the thin plate can in theory be written as an expansion in these time-harmonic solutions. However, the theory for this expansion has only been developed for a maseless-thin plate on shallow water [1]. Here the theory is further developed to water of arbitrary depth.

2 Formulation: A Thin Maseless Plate on Shallow Water

The plate is infinite in the y direction, so that only the x and z directions are considered. The x direction is horizontal, the positive z axis points vertically up, and the plate covers the region $-b \leqslant x \leqslant b$. The water is of uniform depth h. The amplitudes are assumed small enough that the linear theory is appropriate, and the plate is sufficiently thin that the shallow draft approximation may be made. Furthermore the plate is assumed maseless. This means that we only consider the stiffness of the plate not its inertia in calculating the equations of motion.

The mathematical description of the problem follows from [2]. The kinematic condition is

$$\partial_t \zeta = \partial_n \mathbf{G}\phi, \qquad ([2.1])$$

where ϕ is the velocity potential of the water at the water surface and ζ is the displacement of the water surface or the plate (from the shallow draft approximation). $\mathbf{G}$ is the harmonic lifting which solves the boundary value problem

$$\begin{aligned} \Delta\psi &= 0 \\ \psi = \phi&, \quad z = 0 \\ \partial_n \psi = 0&, \quad z = -h \end{aligned}$$

and ∂_n is the outward normal derivative. The equation derived by equating the pressure at the free surface is

$$-\rho g \zeta - \rho \partial_t \phi = \begin{cases} 0, & x \notin (-b, b), \\ D\partial_x^4 \zeta, & x \in (-b, b), \end{cases} \qquad ([2.2])$$

where D is the bending rigidity of the plate per unit length, ρ is the density of water and g is the acceleration due to gravity. At the ends of the plate the free edge boundary conditions

$$\lim_{x \downarrow -b} \partial_x^2 \zeta = \lim_{x \uparrow b} \partial_x^2 \zeta = \lim_{x \downarrow -b} \partial_x^3 \zeta = \lim_{x \uparrow b} \partial_x^3 \zeta = 0 \qquad ([2.3])$$

are applied.

Non-dimensional variables are now introduced. The space variables are non-dimensionalised using the water depth h, and the time variables are non-dimensionalised using $\sqrt{h/g}$. The non-dimensional variables are

$$\bar{x} = \frac{x}{h}, \ \bar{t} = t\sqrt{\frac{g}{h}}, \ \bar{\zeta} = \frac{\zeta}{h}, \ \text{and } \bar{\phi} = \frac{\phi}{h^2\sqrt{g/h}}.$$

In these new variables, ([2.1]) and ([2.2]) become

$$\partial_{\bar{t}} \bar{\zeta} = \partial_n \mathbf{G}\bar{\phi} \qquad ([2.4])$$

and

$$-\bar{\zeta} - \partial_{\bar{t}}\bar{\phi} = \begin{cases} 0, & \bar{x} \notin (-\bar{b}, \bar{b}), \\ \beta\partial_{\bar{x}}^4 \bar{\zeta} + \gamma \partial_{\bar{t}}^2 \bar{\zeta}, & \bar{x} \in (-\bar{b}, \bar{b}), \end{cases} \qquad ([2.5])$$

where β is

$$\beta = \frac{D}{\rho g h^4}.$$

For clarity the overbar is dropped from now on.

3 The Energy Inner Product

The spectral-theory solution of equations ([2.4]) and ([2.5]) is based on the spectral theory for a self-adjoint operator. We therefore require an inner product in which the operator is self-adjoint. This inner product, since the system is conservative, is derived from the energy. The potential and displacement both contribute to this energy and we combine them in a two component vector, $U(x,t)$, given by

$$U(x,t) = \begin{pmatrix} \phi(x,t) \\ i\zeta(x,t) \end{pmatrix}. \qquad ([3.1])$$

The energy consists of the kinetic energy of the water ($\propto |\phi_t^2|$), the potential energy of the water ($\propto |\phi^2|$), and the energy of the plate. The energy inner product for the two vectors

$$U = \begin{pmatrix} \phi(x,t) \\ i\zeta(x,t) \end{pmatrix} \text{ and } U' = \begin{pmatrix} \phi'(x,t) \\ i\zeta'(x,t) \end{pmatrix}$$

is

$$\langle U, U' \rangle_{\mathcal{H}} = \langle \nabla \mathbf{G}\phi, \nabla \mathbf{G}\phi' \rangle + \langle \zeta, \zeta' \rangle + \chi_P \beta \langle \partial_x^2 \zeta, \partial_x^2 \zeta' \rangle \qquad ([3.2])$$

where χ_P is the characteristic function for the plate. This inner product can also be written as

$$\langle U, U' \rangle_{\mathcal{H}} = \langle \partial_n \mathbf{G}\phi, \phi' \rangle + \langle (1 + \chi_P \beta \partial_x^4)\, \zeta, \zeta' \rangle$$

provided that the functions satisfy the appropriate smoothness and boundary conditions. The subscript $\mathcal{H}$ is used to denote the special inner product and the angle brackets without the $\mathcal{H}$ denote the standard inner product, i.e.

$$\langle f(x), g(x) \rangle = \int_{-\infty}^{\infty} f(x)\, g^*(x)\, dx.$$

We now write ([2.4]) and ([2.5]) as

$$\begin{aligned} \frac{1}{i}\partial_t U &= \mathcal{P}U \qquad [3.3] \\ U(x,t)_{t=0} &= U_0(x) = \begin{pmatrix} \phi_0(x) \\ i\zeta_0(x) \end{pmatrix} \end{aligned}$$

where the operator $\mathcal{P}$ is

$$\mathcal{P} = \begin{pmatrix} 0 & 1 + \chi_P \beta \partial_x^4 \\ \partial_n \mathbf{G} & 0 \end{pmatrix}.$$

The operator is $\mathcal{P}$ self adjoint in the energy inner product ([3.2]). We can express the solution to ([3.3]) as

$$U(x,t) = e^{i\mathcal{P}t} U_0 \qquad ([3.4])$$

where $e^{i\mathcal{P}t}$ is a unitary operator.

4 The Expansion in Eigenfunctions

In this section, a solution for the time dependent motion of the plate-water system is developed using the theory of self-adjoint operators. To evaluate equation ([3.4]) we require a method to calculate the evolution operator $e^{i\mathcal{P}t}$. This can be accomplished by using the eigenfunctions of the operator $\mathcal{P}$, which are the single frequency solutions.

4.1 Finding the eigenfunctions

Since $\mathcal{P}$ is self-adjoint, the eigenvalues, λ, must be real and therefore the eigenfunctions of $\mathcal{P}$ are oscillatory exponentials outside the region of water covered by the plate. Furthermore, since the plate is finite, the spectrum (set of eigenvalues) is the entire real numbers. As is expected for two-component systems, there are two eigenfunctions associated with each eigenvalue λ. We choose incoming waves from the left ($U^>$) and the right ($U^<$) of unit amplitude as a basis for the eigenspace since they are the standard single frequency solutions. $U^{>}$ have the following asymptotics,

$$\begin{aligned} \lim_{x\to-\infty} U^{>}(x,\lambda) &= \begin{pmatrix} e^{\mathrm{sgn}(\lambda)ikx} \\ \lambda e^{\mathrm{sgn}(\lambda)ikx} \end{pmatrix} + S_{11}\begin{pmatrix} e^{-\mathrm{sgn}(\lambda)ikx} \\ e^{-\mathrm{sgn}(\lambda)ikx} \end{pmatrix} \\ \lim_{x\to+\infty} U^{>}(x,\lambda) &= S_{12}\begin{pmatrix} Te^{\mathrm{sgn}(\lambda)ikx} \\ Te^{\mathrm{sgn}(\lambda)ikx} \end{pmatrix} \end{aligned} \qquad [4.1]$$

where k is the positive real solution to the dispersion equation

$$k\tanh k = \lambda^2$$

and S_{11}, $S_{12,}$ S_{21}, and S_{22} are the reflection and transmission coefficients (which must be determined). .

We find the eigenfunctions by solving the following equation

$$\begin{pmatrix} 0 & 1+\chi_P\beta\partial_x^4 \\ \partial_n\mathbf{G} & 0 \end{pmatrix} U = \lambda U,$$

which is the boundary value problem

$$\begin{aligned} \Delta\phi &= 0, \quad -\infty < x < \infty, \quad -H < z < 0 \\ i\lambda\zeta &= \partial_n G\phi, \quad z=0 \\ \lambda\phi &= \left(1+\chi_P\beta\partial_x^4\right) i\zeta. \end{aligned} \qquad [4.2]$$

The solution method used here is based on a boundary element formulation for the water as described in [3].

These eigenfunctions satisfy the following orthogonality conditions

$$\begin{gathered} \left\langle U^{>}(x,\lambda), U^{>}(x,\lambda')\right\rangle_{\mathcal{H}} = 4\pi\lambda^2\delta\left(\lambda-\lambda'\right). \\ \left\langle U^{<}(x,\lambda), U^{<}(x,\lambda')\right\rangle_{\mathcal{H}} = 4\pi\lambda^2\delta\left(\lambda-\lambda'\right). \\ \left\langle U^{>}(x,\lambda), U^{<}(x,\lambda')\right\rangle_{\mathcal{H}} = 0. \end{gathered}$$

The rigorous justification of these formulas involves complicated arguments essentially developed in mathematical physics for scattering theory [4]. By a compactness argument (related to the boundedness of the plate), they are actually deduced from the similar formulas which hold for the unperturbed problem, i.e., without the plate. In this case, the latter simply follow from the use of the horizontal Fourier transform.

We expand the motion as

$$U(x,t) = \int_{-\infty}^{\infty}\left(A(\lambda)U^{>}(x,\lambda) + B(\lambda)U^{<}(x,\lambda)\right)e^{i\lambda t}d\lambda$$

and taking inner product we obtain

$$A(\lambda) = \left\langle U_0(x), U^{>}(x,\lambda)\right\rangle_{\mathcal{H}}/4\pi\lambda^2$$

and

$$B(\lambda) = \left\langle U_0(x), U^{<}(x,\lambda)\right\rangle_{\mathcal{H}}/4\pi\lambda^2.$$

We can make some simplifications to the formulation in the case when the initial potential and displacement correspond to an incoming wave.

5 Results

We will repeat the calculations that were made in [1]. We begin by considering an incoming wave. The plate length is $b = 50$ and stiffness is $\beta = 2\times10^4$. We consider a wave which is incoming from the left with potential given by

$$\phi(x) = e^{-(x+125)^2/350}$$

and with the corresponding displacement so that the pulse is travelling to the right. The potential for the times $t = 0$, 30, 60, 90, 120, 150, 180, 210 and 240 are shown in Figure 1. This figure is identical with the equivalent figure in [1].

We now consider the evolution of the plate released from an initial displacement. We use the same values as before, and the initial plate potential and displacement is given by

$$U_0 = \begin{pmatrix} 0 \\ ie^{-(hx)^2/350} \end{pmatrix}.$$

Figure 2 shows the plate displacement ζ for the times shown. Again this is the same as the equivalent figure in [1]

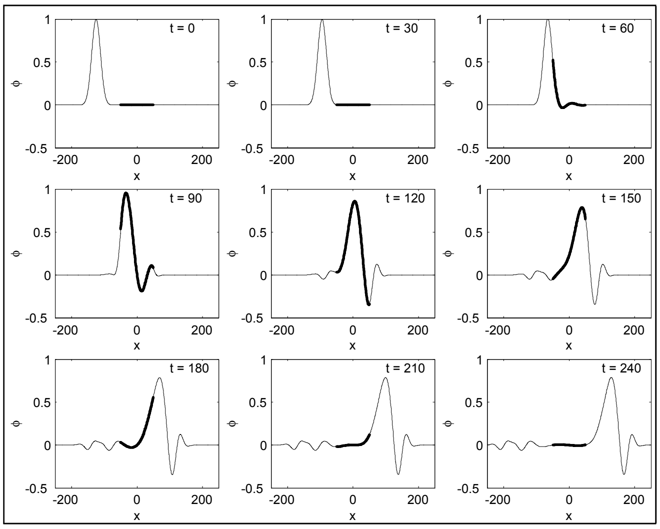

Figure 1. The evolution of the potential due to a pulse travelling to the right for the times shown. The plate occupies the region $-50 \leq x \leq 50$ and is shown by the bold line. $\beta = 2 \times 10^4$, $b = 50$.

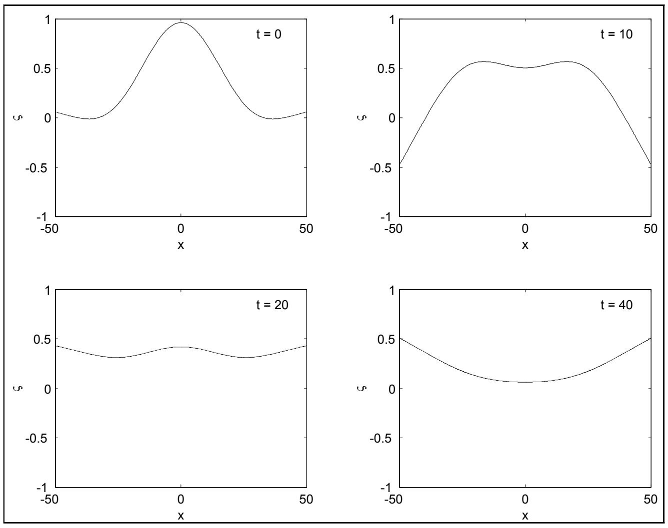

Figure 2. The evolution of a symmetric displacement for a plate released at $t = 0$ for the times shown. $\beta = 2 \times 10^4$, $b = 50$.

6 Conclusions

We have presented a method to calculate the time dependent motion of maseless thin plate floating on the surface of water of arbitrary depth. The method is based on finding an inner product in which the operator describing the plate water system is self-adjoint. The time dependent solution is then found by an expansion in the eigenfunctions of this self-adjoint operator. These eigenfunctions are the time-harmonic solutions.

[1] Michael H Meylan, "Spectral solution of time dependent shallow water hydroelasticity," *Meylan, M. H., Spectral Solution of Time Dependent Shallow Water Hydroelasticity, in press J. of Fluid Mechanics.*, 2002.

[2] J. J. Stoker, *Water Waves: The Mathematical Theory with Applications*, Interscience, 1957.

[3] Michael H. Meylan, "A variation equation for the wave forcing of floating thin plates," *J. of Applied Ocean Res.*, vol. 23, no. 4, pp. 195–206, 2001.

[4] Christophe Hazard, "Analyse modale de la propagation des ondes).," Habilitation thesis, Universite Pierre et Marie Curie., 2001.

Discussion Sheet

Abstract Title :	Spectral theory for a floating massless thin plate on water of arbitrary depth

(Or) Proceedings Paper No. :	32	***Page :***	123

First Author :	Meylan, M.H.
Discusser :	Nikolay Kuznetsov

Questions / Comments :

Is the uniqueness theorem proved for the problem? For proving this theorem, one has to require the energy to be locally finite near the plate's tips. Do you impose this condition?

Author's Reply :
(If Available)

To the best of my knowledge, uniqueness has not been proved for this problem. However, the problem is almost classical so it is possible the result is in the literature. My interest is in using the single frequency solutions rather than their properties (such as uniqueness, or the method used to calculate them). The free plate edge conditions are used at the plate tips.

Discussion Sheet

Abstract Title :	Spectral theory for a floating massless thin plate on water of arbitrary depth		
(Or) Proceedings Paper No. :	32	***Page :***	123
First Author :	Meylan and Hazard		
Discusser :	J. Nick Newman		

Questions / Comments :

Fourier transformation between the frequency and time domains has a long history in this field, including Ursell's analysis of transient motions of a circular cylinder in 1964 and 1970. In order to achieve an accurate transform from the frequency-domain to the time-domain special attention is required regarding the high-frequency asymptotics.

Author's Reply :
(If Available)

The calculation of the high-frequency asymptotics is very important. I have to confess that this importance has only recently become apparent to me and for this reason is not emphasised in my paper. It may well be that the calculation of high frequency asymptotics is critical to the practical application of these Fourier methods.

Discussion Sheet

Abstract Title :	Spectral theory for a floating massless thin plate on water of arbitrary depth		
(Or) Proceedings Paper No. :	32	***Page :***	123
First Author :	Meylan, M.H.		
Discusser :	Ken Takagi		

Questions / Comments :

Is it possible to apply the structural damping in your formulation?

Author's Reply :
(If Available)

Not under the present formulation since energy is no longer conserved. However, it may be possible to extend the theory to this case.

Questions from the floor included; Touvia Miloh & Rod Rainey.