Excitation of trapped modes by the forced motion of structures

P. McIver[1], M. McIver[2], and J. Zhang[3]

[1]Department of Mathematical Sciences, Loughborough University, UK. Email: p.mciver@lboro.ac.uk

[2]Department of Mathematical Sciences, Loughborough University, UK. Email: m.mciver@lboro.ac.uk

[3]Royal School of Mines, Imperial College, London, UK. Email: jie.zhang2@ic.ac.uk

SUMMARY

In the context of the linearised water-wave problem it is known that there are certain structures that support trapped modes. These modes exist in the presence of the fixed structure and are free oscillations of the fluid that do not radiate energy to infinity. Here the consequences of the existence of such modes are investigated. In particular, numerical and asymptotic methods are used in the time domain to show how trapped modes can be excited by the forced motion of a structure.

1 INTRODUCTION

Trapped modes are free oscillations with finite energy of an unbounded fluid for which the fluid motion is essentially confined to the vicinity of a fixed structure; it is important to note that a trapped mode does not radiate energy to infinity. For structures in the open sea, such trapped modes are possible only for particular shapes of structure, and then they occur only for discrete frequencies of oscillation of the fluid. A number of examples of such trapping structures have now been discovered, but here particular attention will be paid to a class of two-dimensional surface-piercing structures first found for a fluid of infinite depth by McIver [1]. Here the equivalent structures in finite depth water will be investigated and a typical example is shown in figure 1 (see §2 for notation).

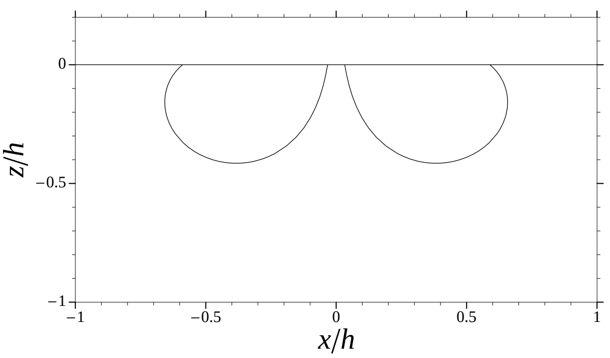

Figure 1: Example of a trapping structure.

In the frequency domain the existence of a trapped mode at a particular frequency implies the non-uniqueness, or sometimes non-existence, of the solution to a linearised scattering or radiation problem when the forcing is at the trapped-mode frequency. If the forcing frequency is away from the trapped-mode frequency then the existence of a trapped mode causes no difficulties in obtaining a solution for the fluid motion at the forcing frequency.

In the time domain, trapped modes may be excited by the forced motion of a structure that is known to support such modes. For example, suppose that the structure is initially at rest and is then displaced in some prescribed way before being brought back to rest. In the linearised problem the fluid motion so generated may be thought of as being made up as individual motions with all frequencies. The motions at frequencies other than the trapped mode frequency will die away as a result of wave radiation to infinity. However, the motion at the trapped frequency is not able to radiate its energy to infinity and, in the absence of viscosity, the fluid oscillation will persist with constant amplitude for all time, even though the structure itself is brought to rest.

In this work the above and other scenarios are investigated in the time domain by using asymptotic methods for large time, and by using a numerical technique. The asymptotic solution is able to predict the amplitude of the trapped-mode oscillation generated from a given structural motion. However, it is unable to describe the initial development of the oscillation and, in particular, it is not able to predict the time taken for the trapped-mode oscillation to become established. Here a numerical method is used for this purpose.

2 FORMULATION

Attention is restricted to two-dimensional problems and Cartesian coordinates (x, z) are chosen with z directed vertically upwards and with the origin in the mean free surface. The fluid domain is bounded below by a flat rigid bed at $z = -h$ but extends to infinity in both the positive and negative x directions. In the linearised time-domain problem, the fluid motion resulting from the motion of an immersed structure may be described by a velocity potential $\Phi(x,z,t)$ that satisfies Laplace's equation and the bed condition

$$\frac{\partial \Phi}{\partial z} = 0 \quad \text{on} \quad z = -h. \tag{1}$$

For a structure Γ that is forced to heave with a vertical component of velocity $V(t)$, the boundary condition to be applied on Γ is

$$\frac{\partial \Phi}{\partial n} = V(t) n_z \tag{2}$$

where n is a coordinate measured normal to Γ and n_z is the z component of the unit normal. The free surface elevation of the fluid $\eta(x,t)$ is related to Φ through the free-surface conditions

$$\frac{\partial \Phi}{\partial t} = -g\eta \quad \text{and} \quad \frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial z} \quad \text{on} \quad z = 0. \tag{3}$$

The motion will be started from rest subject to the initial conditions

$$\Phi(x,0,0) = \frac{\partial \Phi}{\partial t}(x,0,0) = 0 \tag{4}$$

and for any fixed time

$$\nabla \Phi \to 0 \quad \text{as} \quad |x| \to \infty. \tag{5}$$

3 SOLUTION FOR LARGE TIME

A formal solution to the above initial-value problem may be obtained by Laplace transforms. The potential $\Phi(x,z,t)$ is related to its Laplace transform $\widehat{\phi}(x,z,s)$ by the inversion formula

$$\Phi(x,z,t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \widehat{\phi}(x,z,s)\, \mathrm{e}^{st}\ \mathrm{d}s \tag{6}$$

where $s = \gamma$ lies to the right of any poles of $\widehat{\phi}(x,z,s)$. A simple change of variable gives

$$\Phi(x,z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x,z,\omega)\, \mathrm{e}^{-\mathrm{i}\omega t}\ \mathrm{d}\omega \tag{7}$$

where $\phi(x,z,\omega) \equiv \widehat{\phi}(x,z,-\mathrm{i}\omega)$ and the path of integration is taken over any poles of ϕ. It follows from causality that there can be no poles of ϕ in $\mathrm{Im}\,\omega > 0$ and any pole on $\mathrm{Im}\,\omega = 0$ will correspond to the existence of a trapped mode for the particular structure Γ. It will be assumed here that there is only one trapped mode possible for each Γ, that is there are exactly two poles of ϕ for real values of ω at $\omega = \pm\omega_0$, say. This assumption is consistent with the numerical results obtained below.

Suppose that a trapping structure Γ supports a trapped mode at a frequency $\omega = \omega_0$. That is, for this frequency, there exists a non-trivial solution of $\phi_0(x,z)$ of the frequency-domain problem satisfying $\phi_0 \to 0$ as $|x| \to \infty$. For the time-domain problem the forcing velocity in the boundary condition (2) is taken to be of the general form

$$V(t) = a\cos\sigma t + b\sin\sigma t \\ + c\cos\omega_0 t + d\sin\omega_0 t + E(t) \tag{8}$$

where $E(t) \to 0$ as $t \to \infty$. This form includes components both at the trapped-mode frequency ω_0 and at a frequency $\sigma \neq \pm\omega_0$. After some calculation it is found that as $t \to \infty$ the free-surface elevation

$$\begin{aligned}\eta(x,t) \sim A\phi_0(x,0)\Bigg[&\frac{1}{2\omega_0}(d\omega_0 t\sin\omega_0 t \\ &+ c\omega_0 t\cos\omega_0 t + c\sin\omega_0 t) + \frac{1}{\omega_0^2 - \sigma^2}(a\omega_0\sin\omega_0 t \\ &- b\sigma\cos\omega_0 t) + \int_0^\infty E(\tau)\cos\omega_0(t-\tau)\,\mathrm{d}\tau\Bigg] \\ &- \frac{1}{g}\,\mathrm{Re}\Big[\sigma(b - ia)\phi_h(x,0,\sigma)\,\mathrm{e}^{-\mathrm{i}\sigma t} \\ &+ \omega_0(d - ic)\phi_1(x,0)\,\mathrm{e}^{-\mathrm{i}\omega_0 t}\Big].\end{aligned} \tag{9}$$

Here

$$A = -\frac{\displaystyle\int_{-\infty}^{\infty} \phi_0(\xi,-h)\,\mathrm{d}\xi}{\displaystyle\int_F [\phi_0(\xi,0)]^2\,\mathrm{d}\xi} \tag{10}$$

where F denotes the free surface, $\phi_h(x,z,\omega)$ is the frequency-domain heave potential satisfying $\partial\phi_h/\partial n = n_z$ on Γ, and $\phi_1(x,z)$ involves an integral of ϕ_h over frequency. The asymptotic result (9) will be interpreted and compared with numerical results in the following section.

4 NUMERICAL COMPUTATIONS

A numerical method has been used to investigate solutions of the initial value problem described in §2. The method is based on that described by Maiti and Sen [2] in which cubic splines are used to describe variations in both the geometry and the boundary values of the potential Φ and its normal derivative. Time stepping is carried out using the fourth order Runge-Kutta method. The computational domain was truncated at values $x = \pm L$ and a "radiation condition" applied in the form suggested by Clément [3]; this combines rigid pistons at $x = \pm L$ with an absorbing boundary condition in the free-surface section $L - G < |x| < L$. For the computations described here the values $L = 15h$ and $G = 5h$ were used.

The trapping structure shown in figure 1 was used in the calculations. Following McIver [1], this was generated from two oscillatory sources placed in the free surface at $x/h = \pm\pi/8$. For a trapped mode there must be no waves as $x \to \pm\infty$ and this is achieved provided the angular frequency of the oscillation is $\omega_0 = \sqrt{(4\tanh 4)g/h} \approx 1.99933\sqrt{g/h}$. A trapping structure is obtained by locating stream lines of the flow; in this case the particular stream lines emanating from the free surface at $x/h = \pm 3\pi/16$ were used.

In the calculations reported here, the structure is initially at rest with zero displacement and then given a displacement $S(t)$ so that in the boundary condition (2) $V(t) = S'(t)$. In the figures time is scaled by $T = \sqrt{h/g}$.

In the first calculations, shown in figures 2 and 3, $S(t) = \alpha t^3 \,\mathrm{e}^{-t}$ for some constant α (thus $a = b = c = d = 0$ in equations 8 and 9); the factor of t^3 is included to ensure there is no discontinuity in either the velocity or the acceleration as the structure begins to move. For the trapping structures described above, the result of the numerical calculation for the wave elevation $\eta(0,t)$ between the cylinders $\eta(0,t)$ is compared in figure (2) with the asymptotic result (9) which is valid for large time. In the initial phase there is significant wave radiation away from the structure but by about time $t = 10$ the trapped mode is essentially established in isolation and, on the basis of linear inviscid theory, will persist for all time.

To illustrate the behaviour for structures for which no trapped mode exists, $\eta(0,t)$ is plotted in figure 3 for a pair of half-immersed circular cylinders whose intersection with the free surface is the same as the pair of trapping structures shown in figure 1. The imposed velocity is also the same as that for the results given in figure 2. In this case, although a large amplitude fluid motion is set up initially, the amplitude of the motion then decays with time (if rather slowly) as all of the energy of the fluid motion can escape to infinity.

Figures 4–5 are for the trapping structure of figure 1 when subject to a forcing velocity of the form

$$V(t) = \begin{cases} \frac{1}{2}(1-\cos(\pi t/t_m))\alpha\omega\cos\omega t & 0 \le t < t_m, \\ \alpha\omega\cos\omega t & t \ge t_m, \end{cases} \tag{11}$$

which for $t \ge t_m$ corresponds to a displacement $S(t) = \alpha \sin\omega t$. The additional factor in $V(t)$ for $0 \le t < t_m$ ensures that both the velocity and acceleration are continuous at $t = 0$ and $t = t_m$. For the calculations reported here the choice $t_m = 4$ was made.

For figure 4 the forcing frequency σ is equal to the trapped-mode frequency ω_0. This is the resonant case and leading-order term as $t \to \infty$ involves an oscillation with an amplitude that grows in proportion to the time t. In (9) the heave potential ϕ_h is not known explicitly (although it could be found numerically) and hence only the leading-order term is used in the graphical comparison.

For figure 5 the forcing frequency $\sigma = \frac{1}{2}\pi \ne \omega_0$. In agreement with equation (9), for large time the fluid response now contains two oscillatory components, one at the forcing frequency σ and one at the trapped-mode frequency ω_0. The σ component of the asymptotic solution is just the usual frequency-domain solution. The amplitude of the trapped-mode oscillation depends on the initial conditions through $E(t)$. Because the heave potential ϕ_h is not known explicitly the leading-order asymptotic solution is not computed in this case.

5 CONCLUSIONS

The forced oscillations of a structure that supports a trapped mode have been examined in the time domain using asymptotic and numerical methods. Almost any forcing, whether sustained or transitory, will excite the trapped mode and in the absence of friction it persists for all time.

Oscillatory forcing at the trapped-mode frequency produces fluid oscillations with growing amplitude. Oscillatory forcing at a frequency that differs from the trapped-mode frequency gives fluid oscillations with components at both frequencies. This has consequences for the frequency-domain problem in which it is usually assumed that any transients arising from the initial conditions have died away to leave only oscillations at the forcing frequency. For structures that support trapped modes then almost any initial condition will also lead to persistent fluid oscillations at the trapped-mode frequency.

The computations reported above are restricted to two-dimensional surface-piercing structures. Trapped modes are also known to exist in three dimensions and for submerged structures and investigation of these cases will be made in the future.

The fluid response (both for trapping and non-trapping structures) can exceed the displacement of the structure by a considerable amount. Numerical methods can be used to see how this effect is restricted by nonlinearity.

6 ACKNOWLEDGEMENT

This work is supported by the UK Engineering and Physical Sciences Research Council under grant GR/M30937.

7 REFERENCES

1. McIver, M. An example of non-uniqueness in the two-dimensional linear water wave problem. *Journal of Fluid Mechanics*, **315**, 257–266, 1996.
2. Maiti, S., and Sen, D. Nonlinear heave radiation forces on two-dimensional single and twin hulls. *Ocean Engineering*, **28**, 1031-1052, 2001.
3. Clément, A. Coupling of two absorbing boundary conditions for 2D time-domain simulations of free surface gravity waves. *Journal of Computational Physics*, **126**, 139–151, 1996.

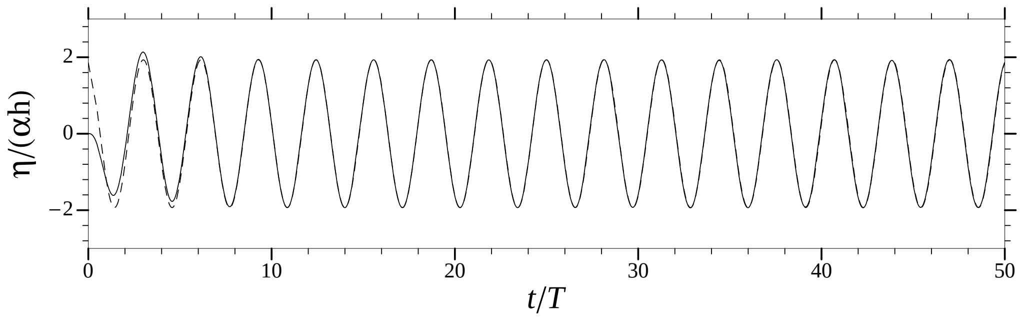

Figure 2: Free surface elevation $\eta(0, t)$ resulting from a displacement $S(t) = \alpha t^3\,\mathrm{e}^{-t}$ of a trapping structure; (——-) numerical calculation, (– – – –) asymptotic solution.

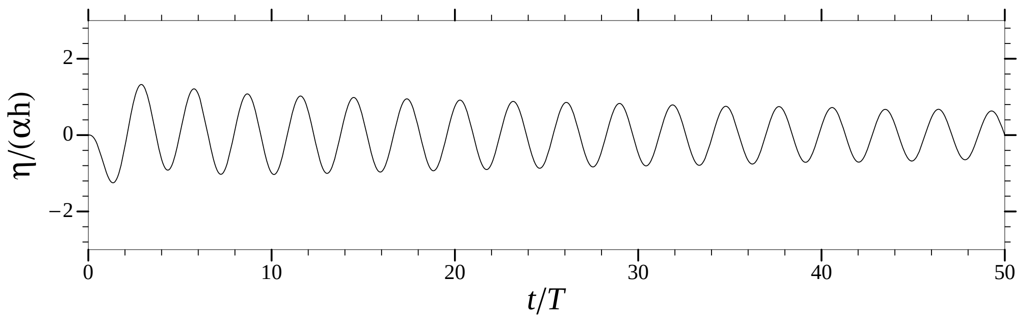

Figure 3: Free surface elevation $\eta(0, t)$ resulting from a displacement $S(t) = \alpha t^3\,\mathrm{e}^{-t}$ of two circular cylinders.

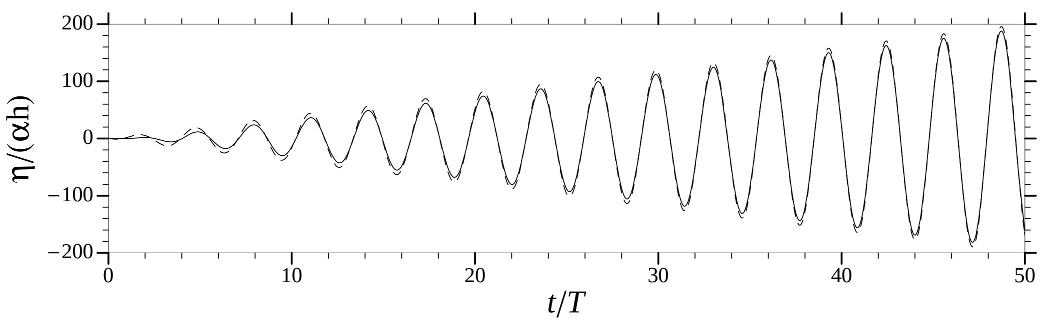

Figure 4: Free surface elevation $\eta(0, t)$ resulting from the oscillatory displacement of a trapping structure at the trapped-mode frequency ω_0; (——-) numerical calculation, (– – – –) asymptotic solution.

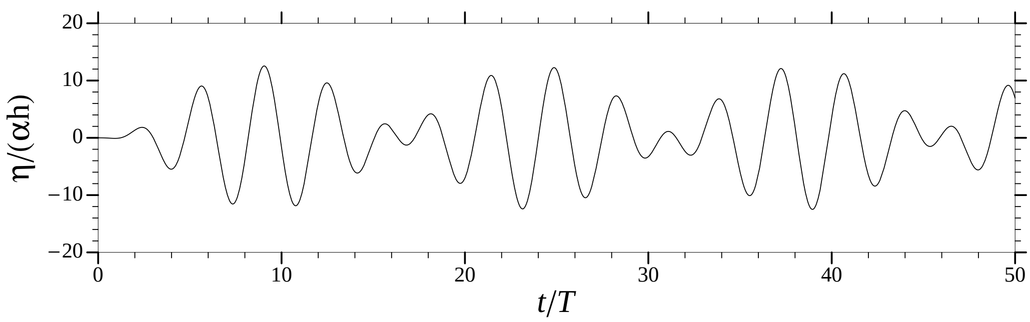

Figure 5: Free surface elevation $\eta(0, t)$ resulting from the oscillatory displacement of a trapping structure at a frequency $\sigma \neq \omega_0$.

17th IWWWFB

Discussion Sheet

Abstract Title :	Excitation of trapped modes by the forced motion of structures		
(Or) Proceedings Paper No. :	31	***Page :***	119
First Author :	McIver, P		
Discusser :	John Grue		

Questions / Comments :

Could your structure serve for extraction of wave power? How would a damping (resistance) modify the resonance? What would be the width of the spectrum?

Author's Reply :
(If Available)

A trapping structure like the ones discussed here would not make a good wave energy device as the band width would be very small (see, J.N. Newman, J. Eng. Math 35, 135-147, 1999).
The effects of damping will be qualitatively the same as in any linear system that exhibits resonance. The "peak" will become finite and be shifted.

Discussion Sheet

Abstract Title :	Excitation of trapped modes by the forced motion of structures		
(Or) Proceedings Paper No. :	31	***Page :***	119
First Author :	McIver, P		
Discusser :	Paul D. Sclavounos		

Questions / Comments :

Have you considered studying the statistics of the wave elevation in the region of trapped waves? Do you expect it to be gaussian in the linear case?

Author's Reply :
(If Available)

We have not studied any wave statistics for this problem.