INTERFACE WAVE SCATTERING BY AN ELLIPTIC ARC

Mridula Kanoria[1]* and B.N. Mandal[2]

[1] Department of Applied Mathematics, Calcutta University,
92, A.P.C. Road,Kolkata -700009,India,e-mail:mridulak@cubmb.ernet.in
[2] PAMU, Indian Statistical Institute,
203 B.T.Rd.,Kolkata-700 035, India, e-mail:biren@www.isical.ac.in
* corresponding author.

SUMMARY

The two-dimensional problem of interface wave scattering by a thin elliptic arc submerged in the lower fluid of the two superposed immiscible homogeneous infinite fluids is investigated here by assuming linear theory by using hypersingular integral equation formulation. Very accurate numerical estimates for the reflection coefficient are obtained and these are depicted graphically against the wave number for various configurations of the elliptic arc and different parameters to illustrate the effect of the presence of the upper fluid. The results for the special cases of a thin vertical plate, thin horizontal plate and circular plate submerged in deep water are obtained and very good agreement with results available in the literature is seen to have been achieved.

1. INTRODUCTION

The method of hypersingular integral equation has been first utilized successfully by Parsons and Martin [1,2] to investigate water wave scattering problems involving thin straight or curved barriers present in deep water, the straight barrier may also be surface piercing. Manadal *et al*[3] used this method to investigate interface wave scattering by a thin vertical barrier submerged in the lower fluid of two superposed immiscible homogeneous fluids. Here we investigate a similar problem wherein an elliptic arc shaped thin plate is present in the lower fluid. As in [2], the problem is formulated in terms of a hypersingular integral equation on the discontinuity of the potential function across the curved plate. This equation is solved approximately and the solution is utilized to obtain very accurate numerical estimates for the reflection and transmission coefficients.

2. FORMULATION

The y-axis is chosen vertically downwards into the lower fluid and the plane $y = 0$ is the position of the interface at rest, the lower fluid is of density ρ_1 and occupies the region $y \geq 0$ while the upper fluid is of density ρ_2 and occupies the region $y \leq 0$. A thin curved plate Γ in the form of the arc of an ellipse with axes along the horizontal and vertical directions, is fully submerged in the lower fluid, and any point $q \equiv (x, y)$ on it has the parametric representation

$$x(t) = a \sin \Theta t, \quad y(t) = d - b \cos \Theta t$$

$$\left(\frac{\alpha}{\Theta} \leq t \leq \frac{\beta}{\Theta}; \Theta = \beta - \alpha \right)$$

where $(a \sin \alpha, b \cos \alpha)$ and $(a \sin \beta, b \cos \beta)$ are the two end points of Γ, $2a, 2b$ being the lengths of the axes. Under the assumption of linear theory and irrotational motion, and subpressing the time harmonic factor $e^{-i\sigma\tau}$, a train of interface waves travelling from the direction of $x = -\infty$ can be represented by the velocity potentials $\phi_j^{inc}(x, y) (j = 1, 2)$ where

$$\phi_j^{inc}(x, y) == (-1)^{j-1} e^{(-1)^j My + iMx} \quad (j = 1, 2)$$

with $M = \frac{1+s}{1-s} K \quad (K = \sigma^2/g, s = \rho_2/\rho_1)$. This train of interface waves is incident on the plate Γ. Let $\phi_j(x, y)$ denote the velocity potentials for the resulting motion in the two fluids $(j = 1, 2)$, then $\phi_j(x, y)$ satisfy the coupled boundary value problem described by

$$\nabla^2 \phi_1 = 0, \quad y > 0, \quad \nabla^2 \phi_2 = 0, \quad y < 0, \tag{2.1}$$

$$\phi_{1y} = \phi_{2y}, \; K\phi_1 + \phi_{1y} = s(K\phi_2 + \phi_{2y}) \;\; \text{on } y = 0, \tag{2.2}$$

$$\phi_{1n} = 0 \;\; \text{on } \Gamma \tag{2.3}$$

$$r^{1/2} \nabla \phi_1 \;\; \text{is bounded as } r \to 0 \tag{2.4}$$

where r is the distance from the submerged edges of Γ,

$$\nabla \phi_1 \;\to 0 \text{ as } y \to \infty, \; \nabla \phi_2 \;\to 0 \text{ as } y \to -\infty \tag{2.5}$$

and

$$\phi_j(x, y) \to \begin{cases} T\phi_j^{inc}(x, y) & \text{as } x \to \infty, \\ \phi_j^{inc}(x, y) + R\phi_j^{inc}(-x, y) & \text{as } x \to -\infty \end{cases} \tag{2.6}$$

where T and R are respectively the unknown transmission and reflection coefficients (complex) which are to be determined.

3. SOLUTION

Proceeding as in [3], a hypesingular integral equation formulation is now obtained. If $f(q)$ denote the discontinuity of ϕ_1 across Γ at the point $q \equiv (x, y)$, then $\phi_1(\xi, \eta)(\eta > 0)$ has the representation

$$\phi_1(\xi,\eta) = \phi_1^{inc}(\xi,\eta) - \frac{1}{2\pi}\int_\Gamma f(q)\frac{\partial G}{\partial n_q}(x,y;\xi,\eta)ds_q \tag{3.1}$$

(3.where the expression of $G(x, y : \xi, \eta)$ is given in [3]. A representation of $\phi_2(\xi, \eta)(\eta < 0)$ in the upper fluid in terms of $f(q)$ can also be obtained, but this is not given here. Use of the boundary condition (2.3) leads to the hypersingular integral equation

$$\frac{1}{2\pi}\,⨎_\Gamma f(q)\frac{\partial^2 G(p;q)}{\partial n_p \partial n_q}ds_q = h(p),\ p(\equiv(\xi,\eta)) \in \Gamma. \tag{3.2}$$

Writing $\xi = a\sin\Theta u, \eta = d - b\cos\Theta u,$, and replacing t, u by $\frac{1}{2}(\frac{\alpha+\beta}{\Theta} + t)$ and $\frac{1}{2}(\frac{\alpha+\beta}{\Theta} + u)$ respectively, the equation (3.2) further reduces to the familiar form

$$⨎_{-1}^{1}\left[\frac{1}{(u-t)^2} + \mathcal{K}(u,t)\right]F(t)dt = H(u), \quad -1 < u < 1 \tag{3.3}$$

where $F(t)$ is related to $f(q)$ and must be such that $F(\pm 1) = 0$, $H(u)$ and $\mathcal{K}(u, t)$ are known bounded functions.

To solve the equation (3.3), $F(t)$ is approximated as

$$F(t) = (1-t^2)^{1/2}\sum_{n=0}^{N} a_n U_n(t) \tag{3.4}$$

where $U_n(t)$ is Chebyshev polynomial of the second kind, and $a_n(n = 0, 1, ..N)$ are unknown complex constants. Proceeding as in [3], $a_n(n = 0, 1, ..N)$ satisfy the linear system

$$\sum_{n=0}^{N} a_n A_n(u_j) = H(u_j) \quad j = 0, 1, ...N \tag{3.5}$$

where $u_j = \cos\dfrac{j+1}{N+2}\pi \quad (j = 0, 1, ...N)$ are collocation points, and

$$A_n(u) = -\pi(n+1)U_n(u) + \int_{-1}^{1}(1-t^2)^{1/2}\mathcal{K}(u,t)U_n(t)dt, \tag{3.6}$$

which can be evaluated numerically for $u = u_j$. The linear system (3.5) has been solved to find $a_n(n = 0, 1, ..N)$ numerically. Having found $a_n(n = 0, 1, ..N)$, R and T are now found by using the infinity conditions (2.6), with x, y replaced by ξ, η, in the representation (3.1). This produces

$$R = -\frac{iM\Theta}{2(1+s)}\sum_{n=0}^{\infty} a_n S_n,\ T = 1 + \frac{iM\Theta}{2(1+s)}\sum_{n=0}^{\infty} a_n S_n^* \tag{3.7}$$

where

$$S_n = \int_1^1 (1-t^2)^{1/2}U_n(t)(b\sin\Theta t' - ia\cos\Theta t')$$

$$e^{-M(y(t')-ix(t'))}dt$$

with $t' = \frac{1}{2}(\frac{\alpha+\beta}{\Theta} + t)$, and S_n^* is the complex conjugate of S_n. Since the identity $|R|^2 + |T|^2 = 1$ holds good always, this is utilized to check the correctness of the numerical estimates for $|R|$ and $|T|$ obtained by using (3.7).

4. NUMERICAL RESULTS

The reflection and transmission coefficients $|R|$ and $|T|$ are computed from (3.7) for various values of different parameters. In the approximation (3.4) twenty terms are taken in most of the computations although for many configurations of the elliptic arc, ten to fifteen terms are sufficient to produce an accuracy of six decimal places. The identity $|R|^2 + |T|^2 = 1$ is always checked for any computation.

To ascertain that the present numerical scheme indeed produces very accurate results, we choose $s = 0, \alpha = 0, \beta = \pi, a/d = 0$ to reproduce the results for a thin vertical plate submerged in deep water obtained earlier by Evans[4]. Similarly choosing $s = 0, \alpha = -\pi/2, \beta = \pi/2, b/d = 0$, results for a thin horizontal plate submerged in deep water are obtained and these coincide with the results of [1].

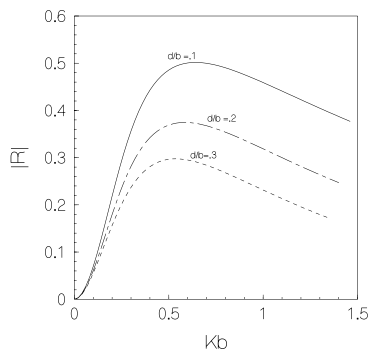

Fig.1:Reflection coeff. vs wave number,a/b=.5,α=0^0,eta=180^0,s=.2

The fig.1 depicts $|R|$ against Kb for a half ellipse $(\alpha = 0, \beta = \pi)$ with $a/b = 0.5, s = 0.2$ for different values of depth $(d/b = 0.1, 0.2, 0.3)$. It is observed that $|R|$ decreases as the depth increases, which is plausible.

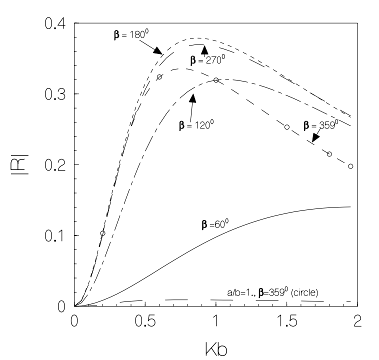

Fig. 2: Reflection coefficient vs wave number,a/d =.2, b/d=.8, α=0^0
0 - estimated from Porter[5]

In fig.2 $|R|$ is plotted against Kb for different arc lengths ($\alpha = 0$, and $\beta = \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{3\pi}{2}, \frac{359\pi}{180}$) in the absence of the upper fluid (s=0) and for $a/d = 0.2, b/d = 0.8$. It is observed that as the arc length increases, the overall value of $|R|$ increases until the arc assumes a half ellipse $(\beta = \pi)$. As the arc length further increases, $|R|$ decreases rather slowly. When it becomes almost a full ellipse $(\beta = 359\pi/180)$, the results for $|R|$ almost coincide with the results obtained by Porter[5] for a submerged ellipse, depicted in the same figure by circles, by using a different technique. It is interesting to note that for $a/b = 1$, the almost full ellipse becomes an almost full circle and there is practically no reflection for all wave numbers as the corresponding curve for $|R|$ suggests. In the presence of the upper fluid almost similar qualitative behaviour of $|R|$ is observed, although the corresponding curves for $|R|$ are not depicted here. Thus, even in the presence of an upper fluid, a circular cylinder submerged in the lower fluid experiences no reflection. Linton and McIver [6] earlier confirmed this when the upper fluid is of finite height and has a free surface.

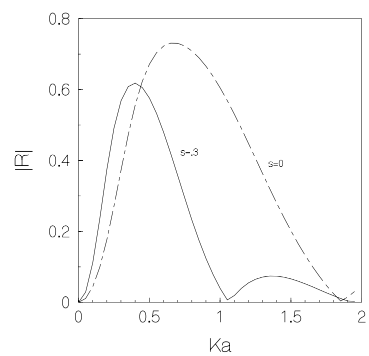

Fig.3:Reflection coeff. vs wave number,a/d=2.,b/d=.1,α=-90^0,β=90^0

In Fig. 3, $|R|$ is depicted against Ka for a half ellipse which is convex upwards $(\alpha = -\pi/2, \beta = \pi/2)$for $a/d = 0.2, , b/d = 0.1$ and $s = 0, .3$. This figure shows that the presence of the upper fluid quickens the occurrence of the first zero of $|R|$ (other than $Ka = 0$) regarded as a function of Ka. Fig. 4 displays $|R|$ against Ka for similar elliptic arc with $b/d = .1$ and different $a/d(2, 4, 6, 8)$ and $s = 0$. As a/d increases the first peak value of $|R|$ increases. For $a/d = 4$, the first peak value is unity near $Ka = .5$. As a/d further increases, a lower peak arises between the two higher peaks (unit values) of $|R|$. This type of behaviour of $|R|$ was also seen in the results of [2] for circular arcs whose arc lengths are kept fixed but radius is increased. In the presence of the upper fluid $(s \neq 0)$, similar qualitative behaviour of $|R|$ for upward convex half ellipse is observed, which is depicted in fig. 5 for similar arcs with fixed a/d, but $b/d = 1, 0$ and $s = 0, .3$. The case $b/d = 0, s = 0$ corresponds to a horizontal thin plate submerged in deep water and the curve for $|R|$ for this configuration is compared with [1], plotted in the same figure by cross marks. Complete agreement is achieved.

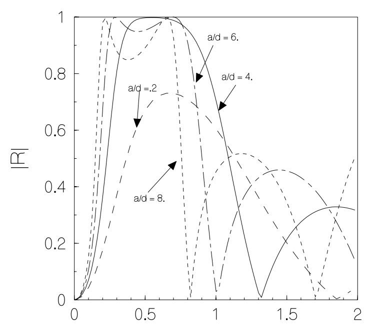

Fig.4: Reflection coefficent vs wave no., b/d=.1,α=90^0,β=90^0,s=0

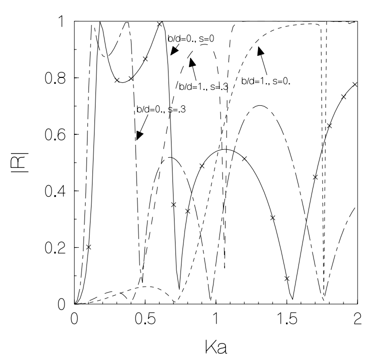

Fig.5: Reflection coeff. vs wave number, a/d=10.,α=-90^0,β=90^0
x - estimaated from Parsons and Martin[2]

5. CONCLUSION

The method of hypersingular integral equation is employed to study the problem of interface wave scattering by an elliptic arc shaped thin plate submerged in the lower fluid of two superposed fluids. The numerical results are illustrated graphically. The effect of the upper fluid reduces the overall effect of the reflection coefficient to some extent. In the absence of the upper fluid, known results for a circular cylinder, thin vertical and horizontal plates submerged in deep water are produced.

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Discussion Sheet

Abstract Title :	Interface wave scattering by elliptic arc		
(Or) Proceedings Paper No. :	18	***Page :***	069
First Author :	Kanoria, M		
Discusser :	Tatiana Khabkhpasheva		

Questions / Comments :

As far as I know, in the diffraction problem of the circular cylinder under an interface we can renormalize forces on the cylinder and the lengthscale in such a way, that the data for the cylinder in a two-layer and in a one-layer fluid will be identical. I believe that in your case it is also possible.

Author's Reply :
(If Available)

Thank you for your suggestion, but what I feel is that if the data is identical then there is no need of doing the same for the two-layered fluid

Discussion Sheet

Abstract Title :	Interface wave scattering by elliptic arc		
(Or) Proceedings Paper No. :	18	***Page :***	069
First Author :	Kanoria, M		
Discusser :	Chris Linton		

Questions / Comments :

In the paper cited by Linton & McIver (1995), we included the effect of an upper free surface and observed the exchange of energy between interfacial and free surface modes.
Presumably, with a change of Green's function, you could consider this case also.

Author's Reply :
(If Available)

Thank you for your suggestion. I will try to do the same.

Questions from the floor included; Masashi Kashiwagi & Guo Xiong Wu.