Embedded trapped modes near an indentation in an open channel

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SUMMARY

We investigate the existence of embedded trapped modes near an indentation in an open channel of uniform water depth. Modes are sought which are symmetric about the centreline of the guide and below the first nonzero cut-off for symmetric wave propagation. An eigenfunction expansion for the trapped mode potential is obtained. A crude approximation is obtained by drastically truncating the eigenfunction expansion and a transcendental equation for the trapped mode frequency is obtained. A full numerical solution is then developed by applying a Galerkin approach. Results show that the approximate solutions are very close to the full solutions. For a given depth of indentation, embedded trapped modes can be found for a series of discrete values of the length of indentation and the wave frequency.

1. Introduction

Trapped mode problems in open channels have been investigated by a number of authors[1]-[3]. Evans and Linton[1] investigated trapped modes in open channels in the presence of either a rectangular block or an indentation on the walls. They found trapped modes which are antisymmetric about the centreline of the channel and which are below the first cut-off for antisymmetric propagation down the guide. Recently, McIver, Linton and Zhang[2] found trapped modes in two-dimensional waveguides in the presence of rectangular block at frequencies which are between the first and second cut-offs. The purpose of this work is to find possible symmetric embedded trapped modes near an indentation in open channels for which the frequency is below the first non-zero cut-off for symmetric wave propagation down the guide. A crude approximation is first exploited and then a Galerkin approach is used to find full solutions.

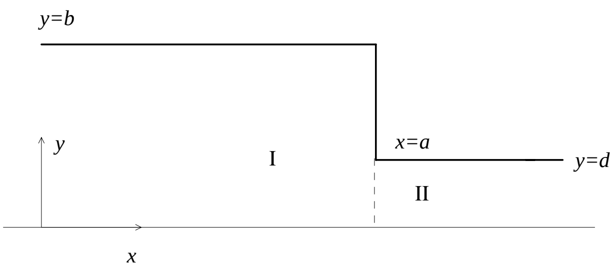

Figure 1: Definition sketch.

2. Formulation

Figure (1) illustrates a quarter section of an open channel. Cartesian coordinates are chosen with the (x, y)-plane in the undisturbed free surface and z vertically upwards. The sides of the channel are at $|y| = d, -\infty < x < \infty$ and the water is of uniform depth H. An indentation symmetrically occupies the region $|x| \le a,\ |y| \le b,\ -H \le z \le 0$ so that it is uniform throughout the entire depth. We define the inner region $0 \le x \le a$ as region I and the outer region $x > a$ as region II. The usual linearised water-wave equation governing the motion of the fluid can be described by a velocity potential $\Phi(x, y, z, t)$ which, assuming harmonic angular frequency ω can be written as

$$\Phi(x, y, z, t) = Re\{\phi(x, y) \cosh k(z + H)e^{-i\omega t}\} \quad (1)$$

where k is the unique positive root of $\omega^2 = gk \tanh kH$ and $\phi(x, y)$ satisfies

$$(\nabla^2 + k^2)\phi = 0 \quad \text{in the fluid.} \quad (2)$$

$$\phi_y = 0,\ |y| = b, |x| \le a;\ |y| = d, |x| > a \quad (3)$$

$$\phi_x = 0,\ |x| = a,\ d < |y| < b \quad (4)$$

$$\phi \to 0,\ |x| \to \infty, |y| \le d \quad (5)$$

Suppose that motion is symmetric about the centreline and $kd < \pi$ but $\pi < kb < 2\pi$. If the indentation is long then the trapped mode must represent waves incident from the left which are totally reflected near $x = a$. This is physically plausible because the range of kd and kb means that there are two possible types of progressive waves in region I but only one in region II. Thus a wave e^{ikx} incident from the left produces a transmitted wave T_1e^{ikx} and a wave $e^{i\alpha x} \cos \pi y/b,\ \alpha^2 + \pi^2/b^2 = k^2$ produces a transmitted wave T_2e^{ikx}, and so a suitable combination of these waves produce no transmission.

We seek non-trivial solutions of (2)-(5) for certain discrete values of kd corresponding to trapped modes. The velocity potential which satisfies (2)-(5) can be either symmetric or anti-symmetric about the plane $x = 0$. Here we restrict our discussion to the potential which is symmetric about both plane $x = 0$ and $y = 0$. The potential can be written in region I and II as

$$\phi_I = \sum_{n=0}^{\infty} U_n^I \frac{\cosh(k_n x)}{k_n \cosh(k_n a)} h_n^I(y) \quad (6)$$

where $k_n = (\lambda_n^2 - k^2)^{1/2},\ 0 \le x < a,\ 0 < y < b$, and

$$\phi_{II} = \sum_{n=1}^{\infty} U_n^{II} \frac{e^{-j_n(x-a)}}{-j_n} h_n^{II}(y) \quad (7)$$

where $j_n = (\mu_n^2 - k^2)^{1/2},\ a \le x < \infty,\ 0 < y < d$, and U_n^I, U_n^{II} unknown coefficients. Functions $h_n^I(y), h_n^{II}(y)$ form orthonormal sets which are given by

$$h_n^I(y) = b^{-1/2}\varepsilon_n \cos \lambda_n y \quad (8)$$

where $\lambda_n = \frac{n\pi}{b},\ n \ge 0;\ \varepsilon_0 = 1;\ \varepsilon_n = 2^{1/2},\ n \ge 1$, and

$$h_n^{II}(y) = d^{-1/2}\varepsilon_n \cos \mu_n y \quad (9)$$

where $\mu_n = \frac{n\pi}{d},\ n \ge 0;\ \varepsilon_0 = 1;\ \varepsilon_n = 2^{1/2},\ n \ge 1$.

The coefficient of the term corresponding to $n = 0$ in (7) is forced to be zero as it represents a progressive wave.

Note that we need two wave like terms in region I and no wave like terms in region II. This means that k_0, k_1 should be imaginary, k_n real for $n \ge 2$, and j_n real for $n \ge 1$. The non-dimensional indentation depth b/d must be in the range of $1 < b/d < 2$ if we restrict $\pi/b < k < \pi/d$.

Continuity of potential and its horizontal derivative across $x = a$ gives

$$\sum_{n=0}^{\infty} \frac{U_n^I}{k_n} h_n^I(y) = \sum_{n=1}^{\infty} U_n^{II} \frac{1}{-j_n} h_n^{II}(y),\ 0 \le y \le d \quad (10)$$

$$\sum_{n=0}^{\infty} U_n^I \tanh(k_n a) h_n^I(y) = \begin{cases} \sum_{n=1}^{\infty} U_n^{II} h_n^{II}(y) & 0 \le y < d \\ 0 & d < y \le b \end{cases} \quad (11)$$

Multiplication of both sides of (10) by $h_m^{II}(y)$, (11) by $h_m^I(y)$, $m = 0, 1, 2, ...$ and integration over $[0,\ d]$, $[0,\ b]$ respectively will give

$$\sum_{n=0}^{\infty} \frac{U_n^I}{k_n} c_{nm} = \begin{cases} -\frac{U_m^{II}}{j_m}, & m \ge 1 \\ 0, & m = 0 \end{cases} \quad (12)$$

$$\tanh(k_m a) U_m^I = \sum_{n=1}^{\infty} U_n^{II} c_{mn},\ m \ge 0 \quad (13)$$

where

$$c_{nm} = \int_0^d h_n^I(y) h_m^{II}(y) dy = \begin{cases} \frac{(-1)^m \varepsilon_m \varepsilon_n \lambda_n \sin(\lambda_n d)}{(bd)^{1/2}(\lambda_n^2 - \mu_m^2)}, & \lambda_n \ne \mu_m \\ (b/d)^{1/2}, & \lambda_n = \mu_m \end{cases} \quad (14)$$

As $c_{0n} = 0$ equation (13) reduces to $\tanh k_0 a = 0$ when $m = 0$ which yields $\tan ka = 0$ since $k_0 a = ika$ and $\tanh(ika) = i \tan ka$. Thus the first condition for a non-trivial solution to exist is

$$ka = n\pi,\ n = 1, 2, 3, ... \quad (15)$$

which is called side condition for convinience.

3. Approximate solution

A crude approximation to these equations is to truncate the series at two terms in (6) and one term in (7). After some algebraic manipulations we have

$$\tan k'a - \frac{4b/d \sin^2(d/b\pi)\ j_1 d}{\pi^2(1 - (b/d)^2)^2\ k'd} = 0 \quad (16)$$

where $k'a = ka(1 - \pi^2/k^2b^2)^{1/2}$, $k'd = kd(1 - \pi^2/k^2b^2)^{1/2}$, and $j_1 d = (\pi^2 - k^2d^2)^{1/2}$.

This is a transcendental equation for kd for a given b/d and $ka = n\pi,\ n = 1, 2, 3, ...$ Solutions of this equation will be compared to those from a full numerical solution.

4. A full numerical solution with Galerkin approach

Following the method described by Evans and Fernyhough[4], we give a brief discription of the Galerkin approach for this specific problem. The common boundary of region I and II is denoted by $\{L:\ x = a,\ 0 \le y \le d\}$, from (11), we write

$$U(y) = \sum_{n=0}^{\infty} U_n^I \tanh(k_n a) h_n^I(y) = \begin{cases} \sum_{n=1}^{\infty} U_n^{II} h_n^{II}(y) & 0 \le y \le d \\ 0 & d \le y \le b \end{cases} \quad (17)$$

It follows that

$$U_n^I = \coth(k_n a) \int_L U(y) h_n^I(y) dy \quad (18)$$

$$U_n^{II} = \int_L U(y) h_n^{II}(y) dy \quad (19)$$

for $n = 1, 2, 3, ...$ since $\tanh(k_0 a)$=0. Substitution of (18) and (19) into (10) gives

$$\int_L U(y') \left\{ \sum_{n=1}^{\infty} \frac{\coth(k_n a)}{k_n} h_n^I(y) h_n^I(y') \right\} dy' + \int_L U(y') \left\{ \sum_{n=1}^{\infty} j_n^{-1} h_n^{II}(y) h_n^{II}(y') \right\} dy' = 0 \quad (20)$$

This is a homogeneous integral equation for $U(y)$. The oscillatory first term in the first summation is shifted to the right hand side. by defining $U(y) = \frac{\cot(k'a)}{k'd}U_1 u(y)$, after some algebra manipulation we have

$$\int_L u(y')K(y,y')dy' = h_1^I(y),\ y \in L \tag{21}$$

where

$$K(y,y') = \sum_{n=2}^{\infty} d^{-1}k_n^{-1}\coth(k_n a)h_n^I(y)h_n^I(y') + \sum_{n=1}^{\infty} d^{-1}j_n^{-1}h_n^{II}(y)h_n^{II}(y') \tag{22}$$

and

$$\int_L u(y)h_1^I(y)dy = k'd\tan(k'a) \tag{23}$$

In addition to the side condition that need to be satisfied, namely, $ka = n\pi$, the problem has been reduced to first solving (21) for $u(y)$, for a given set of geometric parameters, and then looking for trapped mode frequency which can be sustained by the given geometry by solving (23). We shall adopt Galerkin approach to the solution of equations (21), (23) which we first write in the operator form

$$\mathcal{K}\, u = h_1^I \tag{24}$$

with

$$(u,\ h_1^I) = \int_L u(y)h_1^I(y)dy = A \equiv k'd\tan(k'a) \tag{25}$$

Rather than solve(21) directly, the Galerkin approach seeks an approximation $u \approx U$ such that

$$(U,\ \mathcal{K}U) = (U, h_1^I),\ \text{and}\ \ \overline{A} = (U, h_1^I) \tag{26}$$

We choose $u(y) = \sum_{n=1}^{N} a_n u_n(y)$ for some $u_n(y)$ and unknown a_n, substitute into (24), multiply by $u_m(y)$ and integrate over L to give

$$\sum_{n=1}^{N} K_{mn}a_n = F_{m1},\ m = 1, 1, 2, ... \tag{27}$$

where

$$K_{mn} = (\mathcal{K}u_n, u_m),\ F_{m1} = (h_1^I, u_m), \tag{28}$$

then $\overline{A} = \sum_{n=1}^{N} a_n F_{n1}$. If (22) is used in (27) we have

$$K_{mn} = \sum_{r=2}^{\infty} d^{-1}k_r^{-1}\coth(k_r a)F_{mr}F_{nr} + \sum_{r=1}^{\infty} d^{-1}j_r^{-1}G_{mr}G_{nr} \tag{29}$$

Where

$$F_{mn} = (h_n^I, u_m) = \int_0^d h_n^I(y)u_m(y)dy,$$
$$G_{mn} = (h_n^I, u_m) = \int_0^d h_n^{II}(y)u_m(y)dy \tag{30}$$

Thus the equation that is used to calculate trapped mode frequencies can be written as

$$\tan(k'a) - \overline{A}/(k'd) = 0 \tag{31}$$

This equation is very similar to (16).

The choice of $u_n(y)$ is guided by the requirements of correct physical behaviour and simplicity of final forms[5]. We expect that at $y = d$, $u(y)(d-y)^{1/3}$ is bounded which can be derived by a simple conformal mapping argument. In order to preserve simple forms for F_{mn}, G_{mn} and hence K_{mn}, we choose

$$u_n(y) = \frac{2n!\Gamma(1/6)b^{1/2}d^{-1/3}}{(-1)^n\sqrt{2}\pi\Gamma(2n+1/3)(d^2-y^2)^{1/3}}C_{2n}^{1/6}\left(\frac{y}{d}\right) \tag{32}$$

where

$$C_n^{\nu}(\cos\theta) = \sum_{r=0}^{n} \frac{\Gamma(\nu+r)\Gamma(\nu+n-r)}{r!(n-r)![\Gamma(\nu)]^2}\cos(n-2r)\theta \tag{33}$$

are the ultra-spherical Gegenbauer polynomials. After some algebra, it can be shown that

$$F_{mn} = \frac{J_{2m+1/6}(n\pi d/b)}{(2n\pi d/b)^{1/6}} \tag{34}$$

$$G_{mn} = \left(\frac{b}{d}\right)^{1/2} \frac{J_{2m+1/6}(n\pi)}{(2n\pi)^{1/6}} \tag{35}$$

5. Numerical results and discussion

We firstly discuss the approximate solution. From equation (16), the first term $\tan(k'a)$ should be positive since the second term in (16) is negative. Thus it gives $m\pi < k'a < \left(m + \frac{1}{2}\right)\pi,\ \ m = 0, 1, 2, ...$ Combining with the side condition($ka = n\pi$), we have

$$\frac{n\pi d/b}{(n^2-m^2)^{1/2}} < kd < \frac{n\pi d/b}{(n^2-(m+1/2)^2)^{1/2}} \tag{36}$$

$$\frac{b}{d}(n^2-m^2)^{1/2} < \frac{a}{d} < \frac{b}{d}(n^2-(m+1/2)^2)^{1/2} \tag{37}$$

where $m < n;\ n = 1, 2, ...,;\ m = 0, 1, 2, ..., n-1$ since $k'a < ka$. From this relation, we know that, as the length of indentation increases (increasing n), the lowest mode frequency tends to $\frac{\pi}{b/d}$.

For a fixed $b/d > 1$ and $ka = n\pi$, the maximum number of possible trapped mode solutions is n. We denote any one of the solutions as (n, m) in which n comes from $ka = n\pi$, and $m = 0, 1,\ ...,\ n-1$. Thus $(n, 0)$ represents

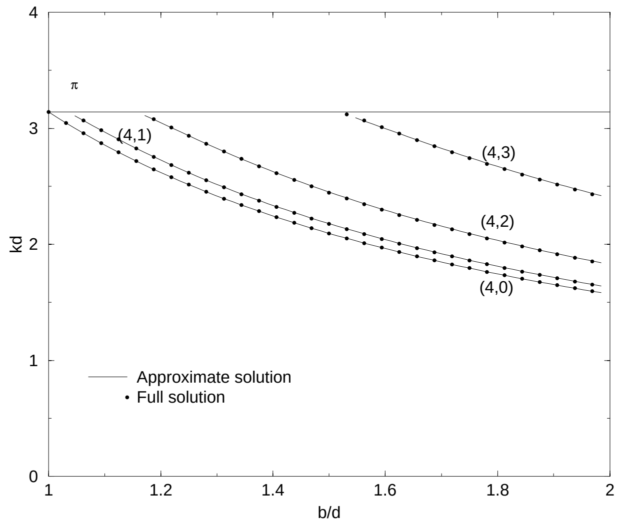

Figure 2: Variation of kd with b/d for $ka = 4\pi$

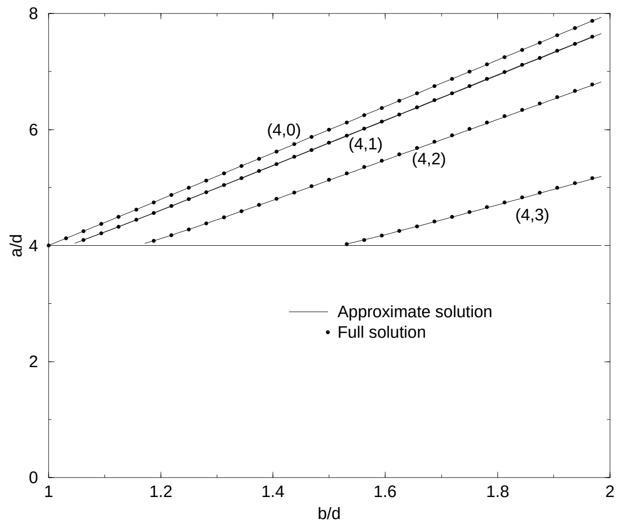

Figure 3: Variation of a/d with b/d for $ka = 4\pi$

the lowest mode and $(n, m-1)$ the highest mode for a given $1 < b/d < 2$. From (36) we can derive the lowest value of b/d above which a higher mode can be found. This can be written as

$$b/d > n/(n^2 - m^2)^{1/2} \tag{38}$$

Thus the lowest mode can always be found if $b/d > 1$ while higher modes can only be found if b/d satisfies (38).

Figure (2) and (3) shows the approximate solution and full solution for $ka = 4\pi$, namely, $n = 4$, where $(4, 0)$ represents the lowest mode and $(4, 3)$ the highest modes. From these results we know that when $b/d \to 1$ the solution tends to be a standing wave solutions, and the approximate solutions are very close to the full solutions. The second mode comes when $b/d > 4/\sqrt{15}$, the third mode $b/d > 4/\sqrt{12}$, and the fourth mode $b/d > 4/\sqrt{7}$.

The use of Gegenbauer polynomials in Galerkin method makes the system converge very quickly. In calculation, 4 decimal place accuracy is achieved by truncating the system at 4 terms.

6. Conclusion

Embedded trapped modes have been found which are symmetric about the centreline of the guide and below the first nonzero cut-off for symmetric wave propagation. For a given depth of indentation, embedded trapped modes can be found for a series of discrete values of the length of indentation and the wave frequency. Results of a crude approximate solution are very close to those from a full numerical solution.

There will be more than two wave-like terms in region I if $b/d > 2$ for $\pi/b < k < \pi/d$. All oscillatory terms in the first summation in (20) should be moved to right hand side. The proper mathematical treatment will result in a system of linear equations for which the condition of non-trivial solution to exist is its determinant to be zero. However, there will be no simple approximate solution availabe in this case.

Acknowledgement

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