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The energy distribution from impact of a three-dimensional body onto a liquid free surface

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1. Introduction

The three-dimensional (3D) problem of blunt-body impact onto a free surface of an ideal and incompressible liquid is considered. During the initial stage of impact the flow region is divided into three parts: (i) outer region, (ii) jet root region and (iii) jet region. In the outer region, the flow is three-dimensional. It is described within Wagner approach. Under the classical assumptions of linearization, this approach leads to a mixed boundary-value problem for the velocity potential in the lower half-space. The boundary of the half-space consists of two parts: the liquid free surface and the wetted surface of the body. These parts are separated by the contact line which varies in time. Position of the contact line is obtained from the so called Wagner condition, which implies continuous joining of the free surface and the surface of the entering body.

Wagner theory is formally valid during initial stage, when the penetration depth of the entering body is much smaller than the dimension of its wetted part. Close to the contact line, the theory fails since both the liquid velocity and the hydrodynamic pressure have singularities along this line. The singularity is yet integrable and the force can hence be calculated. However, in order to get uniformly valid pressure distribution and to improve prediction of the hydrodynamic force on the entering body, a solution which describes details of the flow close to the contact line, must be introduced.

Such a solution was derived by Wagner (1932) in two-dimensional case. The solution was matched with that in the outer flow region. An infinite length of jet was theoretically predicted. For 3D bodies we must even deal with a jet sheet. In order to obtain shape of this jet sheet and the flow inside it, the jet region has to be also considered. The jet solution has then to be matched with that for the jet root region. In the 2D wedge entry problem the jet solution was derived by Howison *et al.* (1991). It was shown that the flow in the jet region is governed by the shallow-water equations and the jet is wedge-shaped. This technique was extended by Korobkin (1994, 1997) to the case of a parabolic contour entering a compressible liquid. Using both the known liquid flow in the jet region and the geometry of this region, the energy of the jet was evaluated in both plane and axisymmetric cases. It was shown that during the impact of two-dimensional or axisymmetric blunt bodies onto a compressible liquid free surface at a constant velocity, half of the work done to move the body goes to the main flow kinetic energy and the other half is taken away with spray jets. The jets are very thin at the initial stage but the jet velocity far exceeds the velocity of the entering body.

This result was confirmed by Molin *et al.* (1996) using another method for 2D problem of impact onto an incompressible liquid surface. This method is based on the concept of energy flux evaluated through the jet root region. The main advantage of this approach is that the flux can be directly determined from the solution in the jet root region and there is no need to deal with the flow in the jet region and its geometry. This approach is used in the present paper to evaluate a part of the energy taken away with the jet in three-dimensional impact problem. It will be shown that, in order to evaluate the jet energy, we need only to know the asymptotic behaviour of the outer solution close to the contact line.

The outer solution for an arbitrary shape of three-dimensional entering body is still not available even within Wagner theory. We restrict the study to elliptic contact line, for which the velocity potential is known and the so called inverse Wagner problem has solutions (see Scolan & Korobkin 2001a). In this frame, shapes of practical interest can be generated (see Scolan & Korobkin 2000).

It is shown that the outer flow – even singular – is approximately two-dimensional close to elliptic contact lines. Therefore it makes it possible to use the planar nonlinear solution by Wagner (1932) for the jet root region. By matching locally the three-dimensional outer solution with the two-dimensional jet root solution, we arrive at a uniformly valid asymptotic description of the pressure distribution. In the case of elliptic contact region this combined solution is used to evaluate the energy distribution throughout the flow domain and to prove that the energy is equally transmitted to the bulk of the fluid and to the spray jet in the case of constant velocity of the entering body.

2. Asymptotics of the outer solution close to the contact line

Within Wagner theory the wetted part of the entering body is approximated by an equivalent expanding flat disc $D(t)$, the boundary conditions are linearised and imposed on the initially undisturbed liquid level

$z = 0$, the liquid flow caused by the impact is assumed irrotational and is described by the velocity potential $\phi_{out}(x, y, z, t)$, where $z < 0$. It is important to notice that the liquid flow in the Wagner approximation depends on both the shape of the contact region $\mathrm{D}(t)$ and the body velocity but not directly on the body shape. We assume that the Wagner problem has been solved already so that the region $\mathrm{D}(t)$ and the body velocity $U(t)$ are prescribed. Moreover, we restrict ourselves to the case of elliptic contact regions, $\mathrm{D}(t) = \{x, y \mid x^2/a^2(t) + y^2/b^2(t) < 1\}$, with the planar and axisymmetric problems representing the limiting cases. Here $a(t)$, $b(t)$ and $U(t)$ are arbitrary positive functions, which satisfy the following inequalities $a(t) \le b(t)$, $U(t) \ll \dot{a}(t)$ and $b(0) = 0$ according to the basic assumptions of Wagner theory. Dot stands for the time derivative.

The velocity potential of the flow initiated by impact of an expanding elliptic disc is given as

$$\phi_{out} = -\frac{Ua^2 b}{E(e)\sqrt{(a^2+\lambda)(b^2+\lambda)}}\sqrt{1 - \frac{x^2}{a^2+\lambda} - \frac{y^2}{b^2+\lambda}} - \frac{Ua^2 bz}{2E(e)}\int_\lambda^\infty \frac{(a^2+b^2+2\sigma)d\sigma}{\sigma^{\frac{1}{2}}(a^2+\sigma)^{\frac{3}{2}}(b^2+\sigma)^{\frac{3}{2}}}, \tag{1}$$

where $e(t) = \sqrt{1 - a^2/b^2}$ and $\lambda(x, y, z, t)$ is the non-negative root of the cubic equation

$$a^{-2}b^{-2}\lambda^3 + L_2\lambda^2 + L_1\lambda - z^2 = 0, \tag{2}$$

$$L_1(x, y, z, t) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{a^2+b^2}{a^2b^2}z^2, \qquad L_2(x, y, z, t) = \frac{1}{a^2} + \frac{1}{b^2} - \frac{x^2+y^2+z^2}{a^2b^2}.$$

This form of the velocity potential is suitable for analysis of the behaviour of the outer solution near the contact line $\Gamma(t) = \{x, y \mid x = a(t)\cos\alpha,\ y = b(t)\sin\alpha,\ 0 \le \alpha < 2\pi\}$.

It is convenient to introduce the local coordinate system (P, x_1, y_1, z_1), where $x = a\cos\alpha + x_1$, $y = b\sin\alpha + y_1$, $z = z_1$ and $(x_1^2 + y_1^2 + z_1^2)/a^2 = O(\varepsilon)$, $\varepsilon \ll 1$. Within the new coordinate system we find

$$L_1 = -2s(x_1, y_1, \alpha, t)[1 + O(\varepsilon)], \qquad s(x_1, y_1, \alpha, t) = x_1 a^{-1}\cos\alpha + y_1 b^{-1}\sin\alpha,$$

$$L_2 = \mu(\alpha, t)[1 + O(\varepsilon)], \qquad \mu(\alpha, t) = a^2(t)\sin^2\alpha + b^2(t)\cos^2\alpha,$$

where $\mu = O(1)$ and $s = O(\varepsilon)$ in the leading order as $\varepsilon \to 0$. Equation (2) provides $\lambda = O(\varepsilon)$ and

$$1 - \frac{x^2}{a^2+\lambda} - \frac{y^2}{b^2+\lambda} = -2s + \frac{\lambda\mu}{a^2b^2} + O(\varepsilon), \qquad \frac{\lambda\mu}{a^2b^2} = s + \sqrt{s^2 + \frac{\mu z_1^2}{a^2b^2}} + O(\varepsilon^2).$$

Therefore,

$$\phi_{out} = -\frac{Ua}{E(e)}\left(\sqrt{s^2(x_1, y_1, \alpha, t) + \frac{\mu z_1^2}{a^2b^2}} - s(x_1, y_1, \alpha, t)\right)^{\frac{1}{2}}[1 + O(\varepsilon^{\frac{1}{2}})]$$

in a small vicinity of the contact line. It is seen that within the coordinate system (P, ξ, η, ζ), which is obtained by rotation of the system (P, x_1, y_1, z_1) counterclockwise at the angle $\theta = \tan^{-1}[(a/b)\tan\alpha]$ so that $x_1 = \xi\cos\theta - \eta\sin\theta$, $y_1 = \xi\sin\theta + \eta\cos\theta$ and $z_1 = \zeta$, the flow is approximately two-dimensional, $s = \xi\sqrt{\mu}/(ab)$, and near the contact line, $\sqrt{\xi^2 + \zeta^2}/a \ll 1$, the velocity potential of the outer flow behaves as

$$\phi_{out} = -UE^{-1}(e)\mu^{\frac{1}{4}}(a/b)^{\frac{1}{2}}(\sqrt{\xi^2 + \zeta^2} - \xi)^{\frac{1}{2}}[1 + O(\varepsilon^{\frac{1}{2}})]. \tag{3}$$

It can be verified that the axes $P\xi$ and $P\eta$ are in normal and tangential directions to the contact line, respectively. Therefore, near the contact line the flow in the tangential direction is negligible compared to the flow in the normal direction. The local flow pattern is given by (3) and is similar to that in the planar case. This two-dimensional flow can be matched to the solution in the jet root region established by Wagner (1932) for the planar impact problem.

3. Parameters of the jet in 3D impact problem

The flow in the jet root region is considered within the moving coordinate system (P, ξ, η, ζ), where $\sqrt{\xi^2 + \eta^2 + \zeta^2}/a \ll 1$ and the local velocity potential ϕ_{root} does not depend on the tangential coordinate η. The flow is approximately quasi-stationary in the leading order as the size of the jet root region tends to zero, and is characterized by the jet thickness $\delta(\alpha, t)$ and the velocity $V(\alpha, t)$ of the fluid in the jet. The dynamic boundary condition shows that the jet velocity $V(\alpha, t)$ is equal to the normal velocity of the point P, which is the origin of the moving system. The 2D jet root solution by Wagner (1932) provides, in particular, the asymptotics of both the velocity potential and the pressure

$$\phi_{root} \approx -4V\sqrt{\frac{\delta|\xi|}{\pi}}, \qquad p_{root} \approx 2\rho V^2\sqrt{\frac{\delta}{\pi|\xi|}}, \tag{4}$$

in the far field, where $|\xi|/\delta \gg 1$, $|\xi|/a \ll 1$ and $\zeta = 0$. Expressions (4) have to be considered as the "outer" asimptotics of the "inner" solution and matched to the "inner" asymptotics (3) of the "outer" solution. Comparing asymptotic formulae (3) and (4), we obtain the jet thickness as

$$\delta(\alpha, t) = \frac{\pi}{8}\frac{U^2(t)(a/b)\mu^{\frac{1}{2}}(\alpha, t)}{E^2(e)V^2(\alpha, t)}. \tag{5}$$

The jet velocity $V(\alpha, t)$ is equal to the normal velocity of the moving contact line, position of which is described by the equation $G(x, y, t) = 0$, where $G(x, y, t) = 1 - x^2/a^2(t) - y^2/b^2(t)$. We obtain

$$V(\alpha, t) = \frac{\dot{G}}{|\nabla G|}, \qquad \dot{G}(\alpha, t) = 2\frac{\dot{a}}{a}\cos^2\alpha + 2\frac{\dot{b}}{b}\sin^2\alpha, \qquad |\nabla G|(\alpha, t) = \frac{2\mu^{\frac{1}{2}}(\alpha, t)}{a(t)b(t)}, \tag{6}$$

where the upper dot denotes the time derivative and ∇ is the gradient operator. Equations (5) and (6) lead to the equality

$$\delta(\alpha, t)V^3(\alpha, t) = \frac{\pi}{16}\frac{U^2a^2}{E^2(e)}\dot{G}(\alpha, t) \tag{7}$$

used below to evaluate the flux of kinetic energy through the jet.

4. Repartition of kinetic energy

It is well-known that the energy conservation law is not satisfied within classical Wagner theory. In general case (see Scolan & Korobkin 2001a),

$$\frac{d}{dt}[A(t) - T(t)] = \frac{1}{2}U^2(t)\frac{dM_a}{dt}, \tag{8}$$

where $T(t)$ is the kinetic energy of the liquid flow in the outer region, $T(t) = \frac{1}{2}M_a(t)U^2(t)$, $A(t)$ is the work done to oppose the hydrodynamic force on the entering blunt body, and $M_a(t)$ is the added mass of the flat disk D(t). During the initial stage of the water impact, the added mass of the expanding flat disk D(t) increases, $dM_a/dt > 0$. Therefore, $T(t) < A(t)$, which is usually considered as an indication that a part of the energy is "lost" during the impact. It is proved below that the flux of energy in the right-hand side of equation (8) is equal to the flux of kinetic energy through the jet in the case of elliptic contact lines.

The total velocity of fluid in the jet $V_f(\alpha, t)$ is equal to the jet velocity $V(\alpha, t)$ plus the normal velocity of the moving contact line, which is $V_f(\alpha, t) = 2V(\alpha, t)$. The part of the kinetic energy $\Delta E_j(\alpha, t)$, which leaves the main flow region through the jet root region of small length $\Delta\ell$ during small time interval Δt, is given as

$$\Delta E_j(\alpha, t) = \frac{1}{2}\Delta m(\alpha, t)V_f^2(\alpha, t), \qquad \Delta m(\alpha, t) = \rho \cdot \delta(\alpha, t)\Delta\ell \cdot V(\alpha, t)\Delta t, \tag{9}$$

where ρ is the liquid density and $\Delta\ell = \mu^{\frac{1}{2}}(\alpha, t)\Delta\alpha$. Equations (7) and (9) provide the total flux of the kinetic energy through the 3D jet in the form

$$\frac{dE_j^{tot}(t)}{dt} = 2\rho\int_0^{2\pi}\delta(\alpha, t)V^3(\alpha, t)\mu^{\frac{1}{2}}(\alpha, t)d\alpha = \frac{\pi}{8}\frac{\rho U^2a^2}{E^2(e)}\int_0^{2\pi}\dot{G}(\alpha, t)\mu^{\frac{1}{2}}(\alpha, t)d\alpha. \tag{10}$$

The integral in (10) is equal to

$$\int_0^{2\pi}\dot{G}(\alpha, t)\mu^{\frac{1}{2}}(\alpha, t)d\alpha = \frac{8}{3ae^2}[(\dot{a}b(1 + e^2) + a\dot{b}(2e^2 - 1))E(e) + (1 - e^2)(a\dot{b} - \dot{a}b)K(e)], \tag{11}$$

where $K(e)$ and $E(e)$ are the complete elliptic integrals of first and second kind, respectively.

The added mass $M_a(t)$ of the elliptic disk D(t) is given as $M_a(t) = 2\pi\rho a^2 b/(3E(e))$, with its time derivative being

$$\frac{dM_a}{dt}(t) = \frac{2\pi\rho a}{3e^2E^2(e)}[(\dot{a}b(1 + e^2) + a\dot{b}(2e^2 - 1))E(e) + (1 - e^2)(a\dot{b} - \dot{a}b)K(e)]. \tag{12}$$

In order to derive equation (12), the following formulae were used

$$\frac{dE}{de} = \frac{E(e) - K(e)}{e}, \qquad \frac{de}{dt} = \frac{a(a\dot{b} - \dot{a}b)}{b^3e}.$$

Substituting (11) into (10) and comparing the result with (12), we obtain

$$\frac{dE_j^{tot}}{dt} = \frac{1}{2}U^2(t)\frac{dM_a}{dt},$$

where the right-hand side is the same as in (8). Equation (8) provides after its integration with respect to time

$$A(t) = T(t) + E_j^{tot}(t).$$

Therefore, the energy conservation law is hold within the 3D Wagner theory if the jet energy is taken into account. It should be noted that this result has been proved only for the case of elliptic contact lines.

It is seen that the energy is equally transmitted to the bulk of the fluid and to the spray jet, $T(t) = E_j^{tot}(t)$, if and only if the velocity of the entering body is constant. If the body velocity is not constant, we find

$$E_j^{tot}(t) = T(t) - \int_0^t M_a(\tau)U(\tau)\dot{U}(\tau)d\tau,$$

where $M_a(\tau) \geq 0$ and $U(\tau) > 0$. Therefore, main part of the energy is transmitted to the bulk of the fluid, $T(t) > E_j^{tot}(t)$, if the body velocity increases, $\dot{U}(t) > 0$, after the impact instant. Correspondingly, the main part of the energy is transmitted to the jet , $E_j^{tot}(t) > T(t)$, if the body velocity decreases, $\dot{U}(t) < 0$, after the impact. The velocity of the entering body decreases, in particular, in the case of free fall of the body onto the liquid free surface.

5. Conclusion

As soon as the matching of the jet root and main flow solutions is performed along the contact line, we know all necessary quantities to evaluate the pressure field. This is done in Scolan and Korobkin (2001b). Two pressure field formulations are considered. Either the composite solution by Zhao and Faltinsen (1992) or the "second order" solution by Cointe (1987) can be used. The force is calculated from the numerical pressure integration all over the wetted area. For the sake of brevity the formulations and results are not presented here. Comparisons are made with results by Zhao and Faltinsen (1997, 1998) which are considered as more exact. For cone and circular paraboloids (as a sphere), it is shown that the "second order" formulation of the pressure provides a force in good agreement for cone aperture less 20^o and before the maximum of force is reached for the sphere. Experimental data concerning the pressure field acting on an elliptic paraboloid should be also obtained soon.

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