Numerical Measurements of the Index of Wave Refraction through a Group of Vertical Cylinders

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1 INTRODUCTION

The interaction of water waves with vertical cylinders has been investigated with special attention in our community those last years. Theses studies were mainly motivated by projects of very large floating structures, like airport, designed to be supported by a huge number of truncated vertical cylinders. When the cylinders are bottom standing, the velocity potential may be found by the semi analytical method of Linton and Evans (1989) [1]. This formulation was used to study some specific phenomenon linked to wave propagation in such regular pile network such as trapped mode [3]. Recently [2] P. McIver, applying the theory and results of solid-state physics to the propagation of water waves in such infinite network of cylinders, showed that the phenomenon of stopping band and passing band may occur also in this hydrodynamic context.

In the present study, the question was to determine if the equivalent of an index of refraction could be defined for the propagation of the water waves through an ocean area occupied by evenly spaced vertical piles when the number of cylinders increases while filling density is kept constant, like in fig.2.

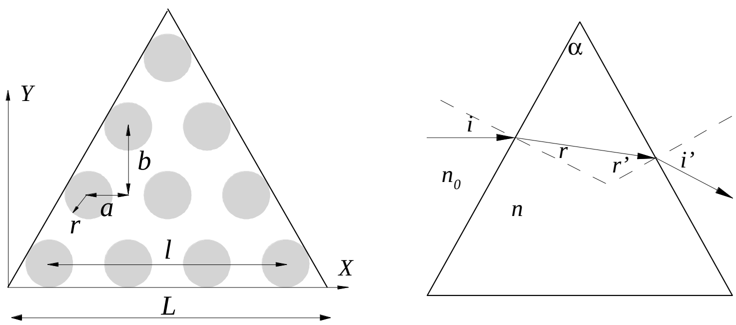

Figure 1: notations. *left*: cylinders filling a triangular area - *right*: refraction of a ray across a prism

To study this question, we have adopted an homogenization approach to the problem, as Evans & Shipway in [4], but using the ray theory of geometrical optics to define an experimental setup for the measurement of the index of refraction. We use the same classical experience as in the study of light propagation through a prism. It is well known that, if the prism medium has an index of refraction of says, n different from the index in the outer open ocean n_0, then the Snell-Descartes law states that:

$$\begin{cases} n_0 \sin i = n \sin r \\ n \sin r' = n_0 \sin i' \end{cases} \quad (1)$$

This law is usually expressed, in linear water wave refraction theory, using the wave celerity instead of the refraction index, reading: $(\sin i)/C_0 = (\sin r)/C$.

A prismatic area (i.e: a triangle view from the top fig.1), filled with cylinders of equal diameters, is exposed to an incident regular wave train described by the usual Airy potential. Two parameters may be used to describe the medium inside the prism: the density d (or *solidity factor* in [4]) which is the ratio of the total cross section area of the cylinders divided by the triangle surface, and the homogeneity factor h which will be defined here as the number of cylinders per unit surface.

Let L be the length of the base of the triangle, l the distance between the extreme circles centers on the base raw, a and b the horizontal and vertical distance between consecutive circles centers, r the radius of the circles (see fig.1). Let p denotes the number of cylinders on the base raw ($p = 4$ in fig.1), then N the total number of cylinders equals $N = \sum_{i=0}^{p-1}(i+1) = p(p+1)/2$. Choosing an equilateral triangle leads to the expression of the density $\sigma = 4\pi N r^2/\sqrt{3}L^2$, and the homogeneity factor $h = 4N/\sqrt{3}L^2$.

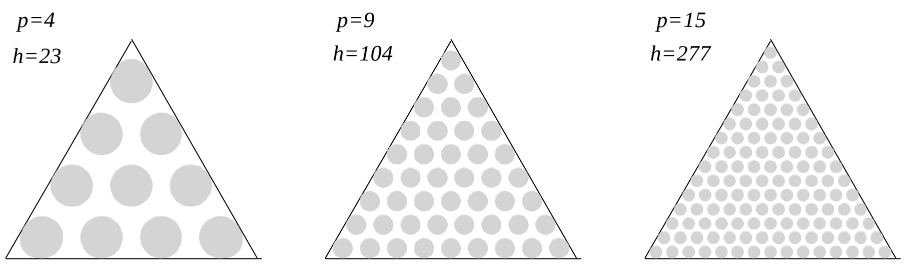

Figure 2: increasing the homogeneity factor h while keeping the density constant: $d = 0.5$

2 NUMERICAL EXPERIMENTS

The numerical experiments consists in first solving the above problem for the potential amplitude by the Linton-Evans method [1], and then by plotting and analyzing the downstream wave field in order to identify, when possible, a transmitted ray refracted of a certain angle to be measured. From this measured angle, the refraction index is derived through the Snell-Descartes law eq.(1).

The computation of the complex potential amplitude follows exactly the method described in [1]. The total potential is then given by

$$\phi = e^{i\kappa r\cos(\theta-\beta)} + \sum_{j=1}^{N}\sum_{n=-\infty}^{\infty} A_n^j Z_n H_n(\kappa r_j) e^{in\theta_j} \tag{2}$$

where we have kept the notations of the cited paper; H_n being the Hankel function of order n, and $Z_n = J_n'(\kappa a)/H_n'(\kappa a)$, with J_n the Bessel function of the first kind and order n. The coefficients A_n^j of the expansion (2) of the scattering potential are the solutions of the system

$$A_m^k + \sum_{j=1}^{N}\sum_{n=-M}^{M} A_n^j Z_n e^{i(n-m)\alpha_{jk}} H_{n-m}(\kappa R_{jk}) = -e^{i(\kappa x_k\cos\beta + \kappa y_k \sin\beta + m\frac{\pi}{2} - \beta)} \tag{3}$$

$$k = 1, \ldots, N \quad m = -M, \ldots, M$$

The first thing we did was to find a range of the parameters (density, homogeneity, wavelength,..) for which the phenomenon can be observed. It is not so evident, because due to other phenomena like partial reflections, medium inhomogeneity, ..., at lot of rays emerge from the triangle, generating confuse refraction figures. Here, for $L = 1$, $d = 0.5$, $\beta = 0$ and $p = 15$, a large range of wavelengths was swept in order to find the best illustrative value. This occurs when the wavelength λ is approximately equal to the horizontal cylinders spacing a.

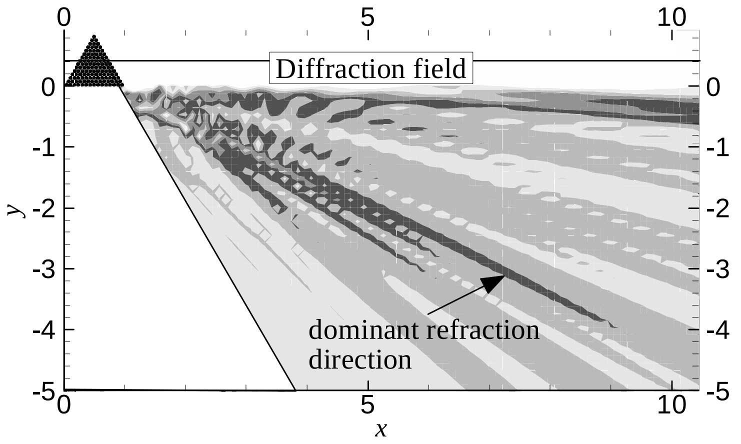

Figure 3: amplitude of the diffraction field showing the refracted ray direction $L = 1$, $d = 0.5$, $p = 15$, $\beta = 0$.

Measurements of the refraction angle are made directly on the plot of the wave field amplitude, in the quadrant where the ray must logically emerge. For the sake of legibility the scattering field is used for this analysis rather than the total wave field. An example of such a plot is given in fig.(3).

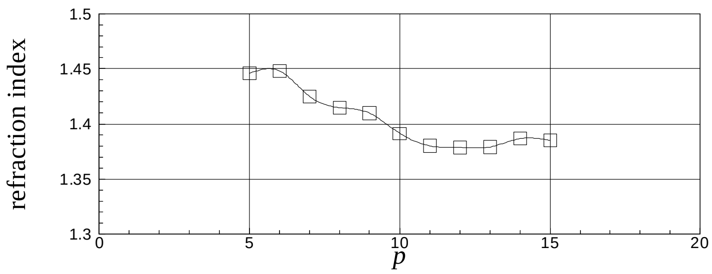

Figure 4: convergence with homogenisation

For this case, the measured refraction index was $n = 1.38$. The convergence to this value with increasing homogeneity is shown in figure (4) where the index is given as a function of the number p of cylinders on the base raw.

The next step was to investigate wether or not the observed phenomenon could be actually attributed to wave refraction. So, from the above case, we varied the incidence angle β and we checked the behavior of the refraction angle with regard to Snell-Descartes law eq.(1).Results obtained when varying the angle of incidence from -10 to 30 degrees are plotted in figure (5). The hydrodynamic results are in good agreement with the optical reference law; we can therefore conclude that the observed deviation phenomenon is most probably of refraction nature, in the common sense.

Finally, a comparison of our results was made with the continuum model proposed by Evans and Shipway at the last Workshop [4]. Their approach was based on an analogy of the equations of the 2D hydrodynamic problem with an acoustic model of the air flow in exchanger tube banks. Following their approach, the ratio between the wave velocity outside and the velocity inside the region filled by cylinders, which is nothing but the refraction index as defined here, should be equal to $n = \sqrt{1+d^2}$. In order to compare our results with this proposal, we have performed a series of numerical experiments at zero incidence, with a fixed wavelength and a fixed homogeneity factor, and we varied the density in the range $[0, 0.5]$. Results are plotted in figure (6). Differences up to 20 percents are observed between these two approaches. It is difficult to conclude about these moderate discrepancies because on one side, our approach suffer from experimental uncertainties and numerical limitations of the number of cylinders, while the homogenisation technique used in the acoustical continuum model is based on several important assumptions.

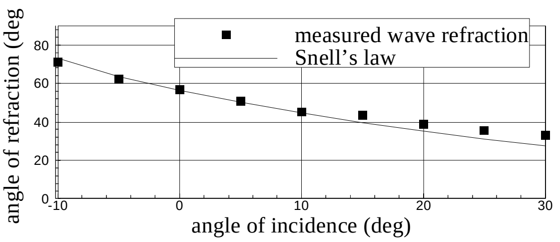

Figure 5: Hydrodynamical versus optical behaviour with varying incidence

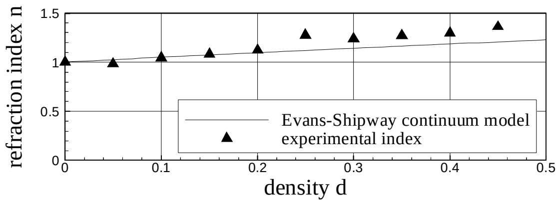

Figure 6: comparison with Evans-Shipway continuum model

3 CONCLUSION

We report here an attempt at verifying the homogenisation approach of wave propagation through an ocean area filled with vertical evenly spaced cylinders proposed by Evans and Shipway at the last Workshop [4]. The basic optical technique of light propagation across a prism is used as a model to analyse water wave propagation across a triangular zone of piles. First of all we had to localize the phenomenon in the whole parameters space; then, for cases where it was clearly observable, we verified that the deviation angle follows the Snell-Descartes law of refraction as expected. The convergence with the homogeneity factor has been tested for several cases, and the measured variation of the refraction index with the pile network density agrees reasonably well with the formula proposed by previous researchers. Nevertheless, it must be pointed out that the phenomenon appeared clearly only for few cases among all our attempts. In the other cases, many spurious rays made the refraction figure indecipherable, probably due to the isotropy of the network and the low level of homogenisation we used here, for computer time economy reasons. The Bragg scaterring phenomenon as described by McIver [2] in this application, could be also the source of unexplained results in some wavelength ranges.

References

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