NUMERICAL SOLUTION FOR UNSTEADY TWO-DIMENSIONAL FREE-SURFACE FLOWS

E. O. Tuck

Department of Applied Mathematics, The University of Adelaide, Australia 5005

An efficient computational procedure is developed for solving fully nonlinear unsteady free-surface problems in two dimensions under gravity. Specifically, we solve here a nonlinear Cauchy-Poisson problem for a flow commencing from rest with a given initial free surface. Our task is thus to solve Laplace's equation $\phi_{xx} + \phi_{yy} = 0$ for the velocity potential $\phi(x, y, t)$, with vanishing velocity at infinity for all finite time t, and two nonlinear boundary conditions on an (unknown) free surface, namely the requirement that it be a material surface, and that the pressure on it be equal to atmospheric.

We use a semi-Lagrangian approach, in which the equation of the free surface is specified parametrically at any time t as $x = X(s, t), y = Y(s, t)$ where s is a marker of a given free-surface point, notionally the initial arclength along the free surface. Then there are three equations determining the free surface evolution, namely (at fixed s, i.e. for each separate free-surface point)

$$\begin{aligned} \frac{dX}{dt} &= u \\ \frac{dY}{dt} &= v \\ \frac{d\Phi}{dt} &= \frac{1}{2}(u^2 + v^2) - gY \end{aligned}$$

where $\Phi(s, t) = \phi(X(s, t), Y(s, t), t)$, and $u = \phi_x, v = \phi_y$.

It is convenient to consider this system of equations as three ordinary differential equations for the functions (X, Y, Φ) of time t, at each fixed free-surface point s. These ODE's can be advanced forward in time by any convenient numerical code, providing the quantities on the right are known at the current time t. If the current potential Φ is known for all s, the tangential velocity can be computed immediately, but not the normal velocity.

At some point in the solution process, the actual Laplace equation must be solved, and this is that point. We therefore must solve a Dirichlet problem with $\phi = \Phi$ assumed known (at the present time t) on the present free-surface boundary, which is assumed temporarily fixed and known. This Dirichlet solution gives $\phi(x, y, t)$ everywhere, and hence allows determination of both velocity components (u, v), so we can proceed to the next time step $t + \Delta t$. During that step the shape of the free surface via $X(s, t + \Delta t), Y(s, t + \Delta t)$, and the value of $\phi = \Phi(s, t + \Delta t)$ on the free surface are updated.

To solve the Dirichlet problem, we simply assume that the velocity potential we require is a sum of N sources, namely

$$\phi(x, y, t) = \sum_{j=1}^{N} q_j \log R,$$

for some to-be-determined source strengths q_j, where R is the distance from the j'th source to the field point (x, y). The sources are located at prescribed positions (X_j, Y_j), the only essential requirement for which is that they lie outside the field of flow, i.e. above the free surface. Given values $\Phi_i = \phi(x_i, y_i)$ on a general surface defined by N given discrete collocation points (x_i, y_i), $i = 1, 2, \ldots, N$, it is then only necessary to solve the linear system of equations

$$\sum_{j=1}^{N} A_{ij} q_j = \Phi_i,$$

for q_j, where

$$A_{ij} = \log \sqrt{(x_i - X_j)^2 + (y_i - Y_j)^2}.$$

The far-field boundary condition can be enforced at this stage by adding a constraint that the total source strength be zero, or more simply by replacing the computed value of the last free-surface height y_N by zero; even more simply, it can be ignored provided the domain extends far enough, and this is the option presently implemented.

In practice, this is a totally straightforward and computationally efficient task, the matrix A_{ij} requiring only a small number of elementary arithmetic operations to compute, and solution for q_j being done by a standard linear equations routine. The velocity components u, v are then also available by summation of elementary functions. This technique is efficient both in programming and computing time, relative to techniques where the singularities are smoothed out over panels.

The only sophistication needed in the technique is in choice of the appropriate source location points (X_j, Y_j), defined by a certain offset distance and direction outside the fluid domain. Most reasonable choices work well for gentle disturbances, but there is value in careful placement of sources for violent free-surface motions. See [2, 3] for further discussion of this matter.

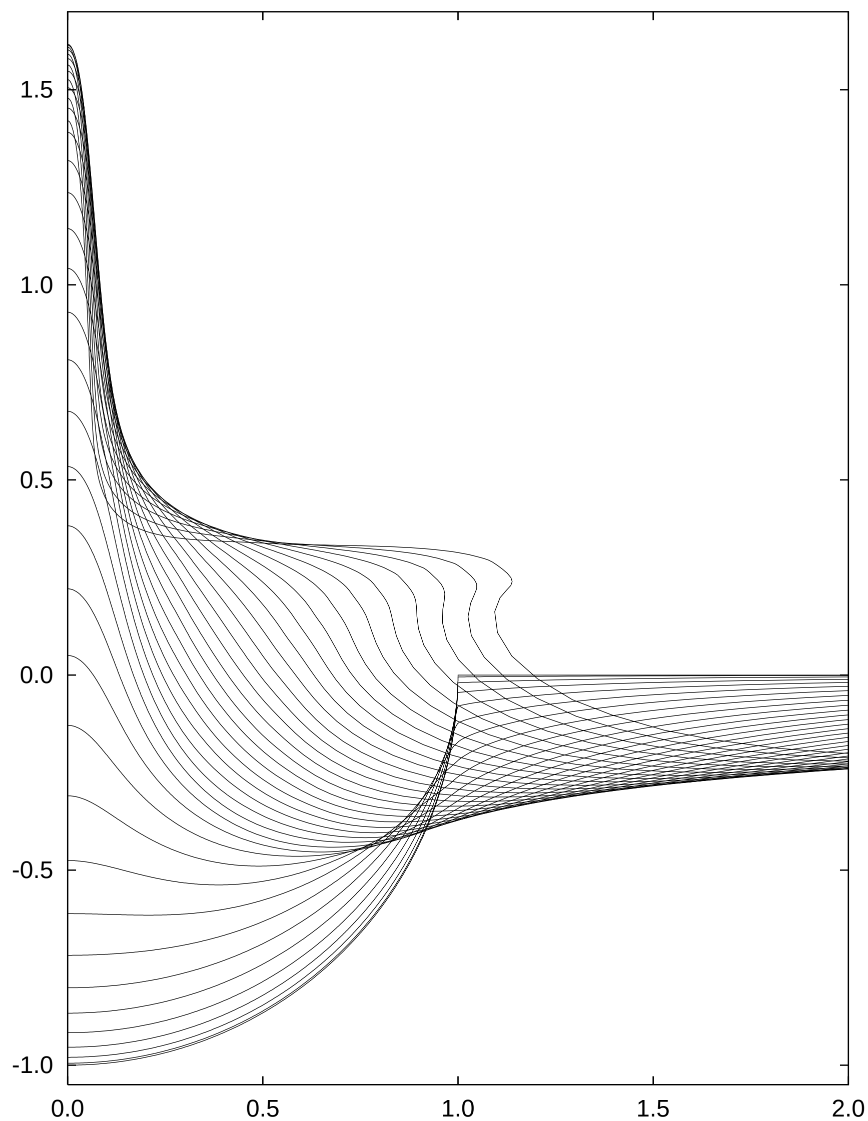

Figure 1: Evolution from rest of an initially semi-circular depression in a plane free surface.

The resulting computational method is very fast, economical and accurate. In particular, it is not necessary to "smooth" the free surface during the course of the computation. Such smoothing seems to have been needed to remove spurious grid-scale oscillations ever since the original work of Longuet-Higgins and Cokelet [1], and recent versions of this algorithm have required smoothing routines of a very high degree of sophistication. In contrast, the present results seem inherently smooth.

Solutions of similar problems for sloshing in a rectangular tank were given in [3], the region of flow being then bounded by two vertical side walls and a horizontal bottom. Any such additional boundaries require a modification of the fundamental source potential $\log R$. For example, plane walls and circular cylinders are easy to accommodate by suitable image sources. Even in the present study for an unbounded domain, it is convenient to assume that the initial elevation and hence the resulting flow is symmetric about $x = 0$, which means that the flow is the same as if there was a vertical wall at $x = 0$. Building in this wall by use of an image source reduces the size of the matrix by half.

Nevertheless we must now allow the flow to extend to $x = +\infty$, which is at first sight potentially more difficult than the finite-domain studies of [3]. However, a simple approach of distributing the required sources and collocation points on a grid that smoothly extends toward infinity seems to work quite satisfactorily. A number of such grids have been tried, but a simple and effective choice is one in which, after assigning half of the points (equally spaced) to the non-zero initial free-surface elevation, the remaining points initially are placed on the x-axis with a spacing that increases geometrically, at present with a ratio of 1.1. For example, with $N = 160$ the furthest point is at about $x_N = 400$. Note that in the present semi-Lagrangian method, these nodal points (and the corresponding source points) move with the flow, and hence change with time. It seems that the more remote sources have little effect on the flow in the finite part of the domain, and their strengths tend to zero quite rapidly.

The first results shown here are for an initial surface displacement similar to that used in [3], namely an initial semi-circular depression of unit radius. Without loss of generality, we can scale $g = 1$. This particular run used $N = 640$ points, with a source offset distance of 3 times the local grid spacing and a fixed Runge-Kutta time step of 0.02, although profiles are shown only at intervals of 0.1, up to $t = 3.6$. Up to that time, the results are the same to graphical accuracy with much lower N (typically $N = 80$), and also with much larger timesteps, and offsets between 2 and 4 times the local grid spacing. Finer grids and time steps are needed to pursue the computations nearer to the breaking crisis which occurs at about $t = 3.7$. Figure 1 shows only the portion of the free surface for $x < 2$; the free surface smoothly approaches zero beyond that point. At the last time shown ($t = 3.6$) the free surface is about to break near $x = 1$. At the earlier time $t = 3.0$, a maximum positive elevation of about $y = +1.6$ has occurred at the central point $x = 0$, and this sharp but smooth spike has begun to fall, remaining well captured by the program until computations are stopped at about time $t = 3.7$ by actual physical breaking near $x = 1$.

Figure 2 shows corresponding results for initially semi-elliptical depressions with drafts of 0.5 and 0.3, computed with $N = 160$. At draft 0.5, the central spike with maximum height of about 0.7 occurring at about time $t = 2.6$ is much sharper (though lower), and this presents difficulties to the present implementation of the program which cause it to fail at about $t = 3.4$, a time when it is not yet possible to tell whether breaking will occur.

On the other hand, at draft 0.3, the program is able to continue essentially indefinitely; results are now shown for $x < 3$ at a plotting interval of 0.4, up to $t = 6.8$. The central spike occurring at about time $t = 2.4$ has reduced to a mere gentle peak, and the main smooth crest subsequently moves off toward infinity without breaking, followed by subsidiary troughs and crests.

These particular configurations are studied because of possible relevance to the flow immediately behind the transom of a slender ship moving at high speed. For example, the outward moving crests in the results for the ellipse with draft 0.3 represent diverging waves in that model.

Acknowledgement: This work was supported by the Australian Research Council. Assistance with the most recent computations and graphical presentations has been provided by Dr David Scullen and Dr Sergey Simakov.

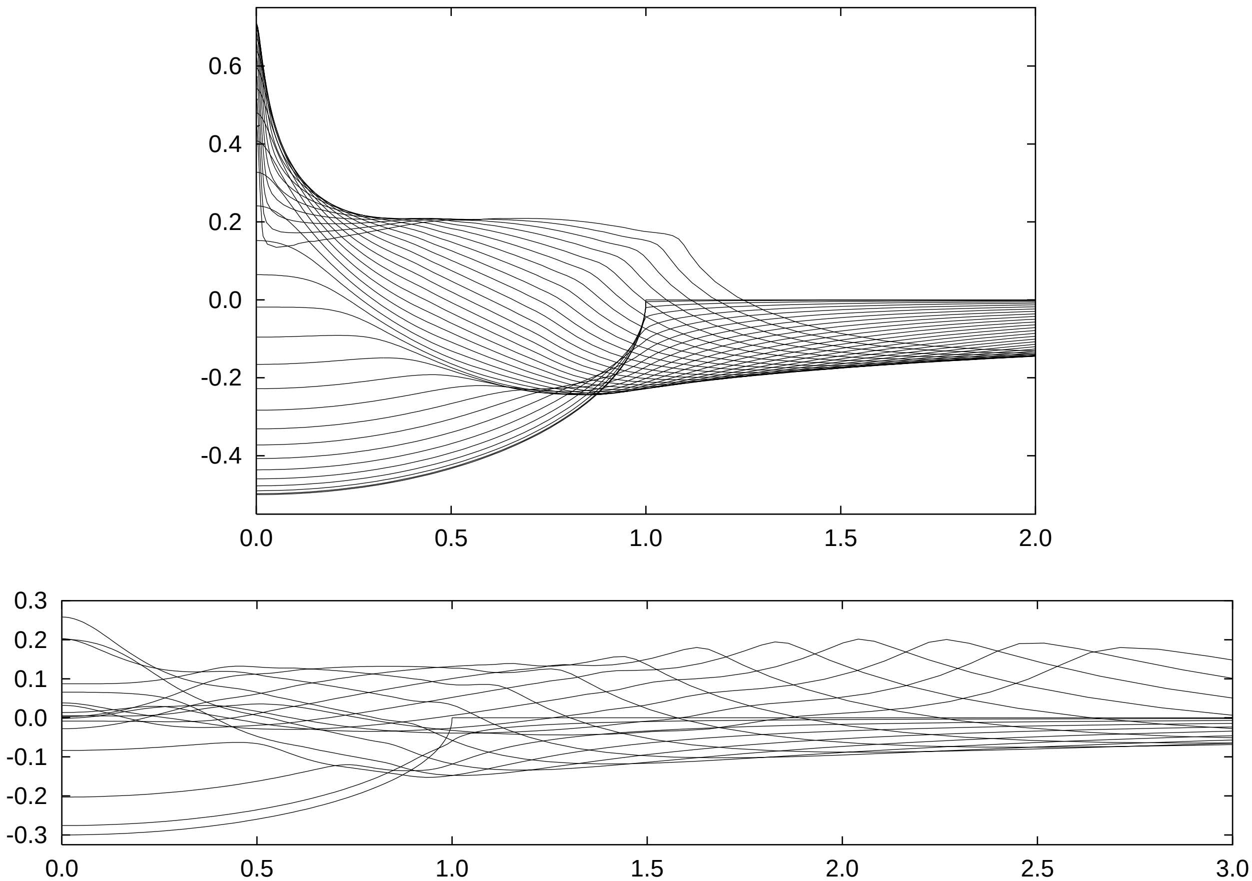

Figure 2: Initially semi-elliptical depressions with drafts 0.5 and 0.3.

References

[1] Longuet-Higgins, M.S. and Cokelet, E.D., The deformation of steep surface waves on water. I. A numerical method of computation, *Proc. Roy. Soc. Lond. Ser. A*, **350**, 1976, 1–26.

[2] Scullen, D. and Tuck, E.O., Nonlinear free-surface flow computations for submerged cylinders, *J. Ship Res.*, **39**, 1995, 185–193.

[3] Tuck, E.O., Solution of nonlinear free-surface problems by boundary and desingularised integral equation techniques, *in Proc. 8th Biennial Computational Techniques and Applications Conference*, ed. J. Noye *et al.*, World Scientific, pp. 11–26, 1997.